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Minimal enclosing discs, circumcircles, and circumcenters in normed planes (Part II) ☆

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ABSTRACT

Until now there are almost no results on the precise geometric location of minimal enclosing balls of simplices in finite-dimensional real Banach spaces. We give a complete solution of the two-dimensional version of this problem, namely to locate minimal enclosing discs of triangles in arbitrary normed planes. It turns out that this solution is based on the classification of all possible shapes that the intersection of two norm circles can have, and on a new classification of triangles in normed planes via their angles. We also mention that our results are closely related to basic notions like coresets, Jung constants, the monotonicity lemma, and d-segments.

1. Introduction

It is well known that the problem to find a minimal (regarding inclusion) circle of a given finite point set goes back to Sylvester [25]. Note that the notion of minimal enclosing ball (see, e.g., [7, §35 and §44] and [24, §14]) of a given point set is closely related to various interesting problems from Convexity and from Discrete and Computational Geometry, such as minimax or 1-center problems (cf., e.g., [9,22,26,11,28,23]), Jung's theorem and Jung constants (see, e.g., [15, §78], [12, p. 49], and [5]), related questions in the spirit of geometric inequalities (see [8, §11]), and also coresets (cf. [2] and [3]). For some of these notions and topics, there is a large variety of references even for finite-dimensional real Banach spaces (here shortly called *normed spaces* or *Minkowski spaces*), but results on the precise geometric location of such extremal balls and their centers are missing, even for the case of normed planes. Closely related references are [14,16–18].

It is our goal to present the first thorough study of all possible locations of minimal (regarding inclusion) enclosing discs for arbitrarily given triangles in general normed planes. This work is the second part of our strongly related paper [1], where the precise location of circumcircles of triangles in arbitrary normed planes is completely studied. As in this paper, our investigations here are also based on the monotonicity lemma for normed planes and on all possible shapes that the intersection of two norm circles can have; see also [4]. But in addition to [1] we need basic notions like that of d-segments (see, e.g., [6, §9]) and, here introduced, norm-acuteness, norm-rightness and norm-obtuseness of triangles in normed planes. Based on this, we present new and partially surprising results on the geometry of triangles in normed planes. More precisely, we give a complete description of the locus of all possible centers of minimal enclosing discs for any norm-dependent

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position of three given points. (This is also summarized in the table at the end of the paper.) In some cases, we describe this locus even via different tools. For example, Theorem 4.1 uses a geometric transform, Theorem 4.2 analytical methods and Corollary 4.1 the Fermat–Torricelli problem from location science to express the same results in this framework. It turns out that, in contrast to the Euclidean situation, there are point triples with infinitely many circumcircles and infinitely many minimal enclosing discs. Even more, we show that there are cases with unique circumcircle, but infinitely many minimal enclosing discs.

2. Preliminaries

Let $(\mathbb{R}^2, \|\cdot\|)$ be a *normed* (or *Minkowski*) *plane*, i.e., a two-dimensional real vector space in which a norm $\|\cdot\|$ is defined. Basic references on the geometry of normed planes and spaces are [27,20], and [19]. Let $\mathcal{D} = \{x \in \mathbb{R}^2\colon \|x\| \leqslant 1\}$ be the *unit disc* of $(\mathbb{R}^2, \|\cdot\|)$, being a compact, convex set centered at the *origin* o, which is interior to $\mathcal{D}$. The boundary of $\mathcal{D}$ is the *unit circle* $\mathcal{C} = \{x \in \mathbb{R}^2\colon \|x\| = 1\}$. A homothetic copy $p + \lambda\mathcal{D}$ of $\mathcal{D}$, where $p \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}^+$, is called the (Minkowskian) *disc* $\mathcal{D}(p,\lambda)$ *with center* p *and radius* λ. Analogously, $p + \lambda\mathcal{C}$ is said to be the (Minkowskian) *circle with center* p *and radius* λ and denoted by $\mathcal{C}(p,\lambda)$. A point of a circle is an *extreme point* if it does not belong to the relative interior of a non-degenerate segment contained in the circle. The plane $(\mathbb{R}^2, \|\cdot\|)$ is *strictly convex* if all the points of $\mathcal{C}$ are extreme points; it is *smooth* if at any point of $\mathcal{C}$ there is a unique supporting line of $\mathcal{D}$.

Due to the triangle inequality the distance between two points belonging to a ball of radius λ is at most 2λ. Thus the minimal radius of a ball containing two points p and q is $\|p-q\|/2$. In contrast to the Euclidean situation, Fig. 1 shows that in an arbitrary normed plane a ball of minimal radius containing two given points may not be unique.

For two different points p, q we will use the symbols $[p,q]$, $[p,q\rangle$, and $\langle p,q\rangle$ for the *line segment*, the *ray* (with starting point p), and the *line* determined by these points. Saying that two non-degenerate *segments are parallel* we mean that their affine hulls are parallel lines. For $p \neq q$ and a point x not from $\langle p,q\rangle$ the closed half-plane bounded by $\langle p,q\rangle$ and containing x will be denoted by $HP_x^+(p,q)$, the opposite one by $HP_x^-(p,q)$.

A point p is said to be *Birkhoff orthogonal* to q, denoted by $p \dashv q$, if $\|p + \lambda q\| \geqslant \|p\|$ for every $\lambda \in \mathbb{R}$, i.e., if the line through p with direction of the vector q supports the circle with center o and radius $\|p\|$ at p.

The *Monotonicity Lemma* will be (sometimes camouflaged) very present along the paper. In the form below it is given in [13]. For more detailed discussions about it we refer to [20, Section 3.5]

Lemma 2.1 *(Monotonicity Lemma). Let $\mathcal{C}$ be the unit circle in a normed plane $(\mathbb{R}^2, \|\cdot\|)$, and p, q, r be different points belonging to $\mathcal{C}$ such that the origin o does not belong to the open half-plane determined by $\langle p,q\rangle$ which contains r. Then*

$$\|p-q\| \geqslant \|p-r\|,$$

with equality if and only if q, r, and $\frac{1}{\|q-p\|}(q-p)$ belong to a segment contained in $\mathcal{C}$.

For three non-collinear points t_1, t_2, t_3 we denote by $T(t_1,t_2,t_3)$ the triangle with these points as vertices. Any circle containing the points t_1, t_2, t_3 in a normed plane is called *circumcircle*, and its center *circumcenter*, of $T(t_1,t_2,t_3)$; and any disc of minimum radius containing these three points is a *minimal enclosing disc* of $T(t_1,t_2,t_3)$. The set of the centers of all the minimal enclosing discs of $T(t_1,t_2,t_3)$ will be denoted by $MEDC(t_1,t_2,t_3)$.

As mentioned in the introduction, minimal enclosing balls are also related to Jung's constant $\mathcal{J}_{\|\cdot\|}$ of $(\mathbb{R}^d, \|\cdot\|)$. Recall that $\mathcal{J}_{\|\cdot\|}$ is the infimum of the $\mu > 0$ such that any set of diameter $\leqslant 1$ can be covered with a ball of diameter μ, i.e., it is two times the supremum of the $\lambda > 0$ that is the radius of a minimal enclosing ball of a set of diameter 1.

Regarding the existence and uniqueness of circumcircles for three non-collinear points t_1, t_2, t_3, the following situations are possible (see, e.g., [20, Propositions 14 and 41]):

1. There exists a unique circumcircle of $T(t_1,t_2,t_3)$. This happens for any t_1, t_2, t_3 if and only if the normed plane is strictly convex and smooth.
2. There exist at least two circumcircles of $T(t_1,t_2,t_3)$. This is only possible when the normed plane is not strictly convex (see Fig. 1).
3. There exists no circumcircle of $T(t_1,t_2,t_3)$. This is only possible when the normed plane is not smooth (see Fig. 2).

It is evident, but important to note, that if we know the radius of the minimal enclosing ball of a set of given points, then we can describe the locus of the centers of all the minimal enclosing balls. Moreover, the cardinality of the given set and the dimension of the space where we are working have no matter for that description. Namely, directly from the definition of minimal enclosing ball it follows that if λ is the radius of a minimal enclosing ball of $\{t_1,\dots,t_n\} \subset (\mathbb{R}^d, \|\cdot\|)$, then $MEDC(t_1,\dots,t_n) = \bigcap_{i=1}^n \mathcal{D}(t_i,\lambda)$. Our task in this paper will be to determine λ and to describe that set for the case of three points in a normed plane.

In Section 3 we extend the notions of acuteness, obtuseness and rightness of the vertices of a triangle in the Euclidean plane to the vertices of triangles in an arbitrary normed plane. This allows to classify the triangles of a normed plane into four classes (see Definition 3.1). In Sections 4 to 6 we give a complete description of the minimal enclosing discs of each

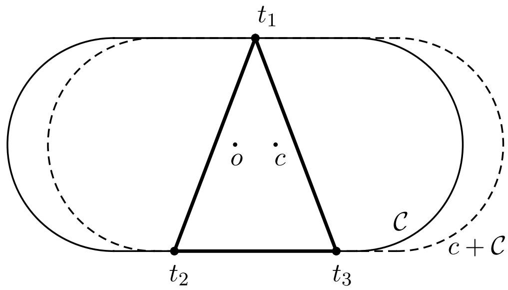


Fig. 1. More than one circle through t_1, t_2, t_3.

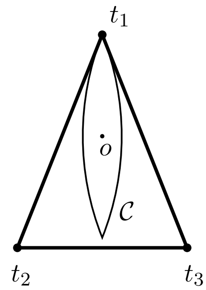


Fig. 2. No circle through t_1, t_2, t_3.

type of triangles based on the different properties they have, these properties being directly related to the existence and/or uniqueness of the corresponding circumcircles.

3. Different types of triangles in normed planes

In analogy to what happens in the Euclidean plane, we say that a triangle $T(t_1, t_2, t_3) \subset (\mathbb{R}^2, \|\cdot\|)$ is *norm-acute* at the vertex t_k if

$$\left\| t_k - \frac{t_i + t_j}{2} \right\| > \frac{\|t_i - t_j\|}{2},$$

where $\{i, j, k\} = \{1, 2, 3\}$. Similarly, we say that $T(t_1, t_2, t_3)$ is *norm-obtuse* (respectively, *norm-right*) at t_k if the above inequality is changed into "$<$" (resp., "$=$").

Proposition 3.1. *Any triangle $T(t_1, t_2, t_3)$ in $(\mathbb{R}^2, \|\cdot\|)$ with $\|t_1 - t_2\| > \|t_2 - t_3\|$ is norm-acute at t_1.*

Proof. The proof follows from the inequalities

$$\left\| t_1 - \frac{t_2 + t_3}{2} \right\| = \left\| t_1 - t_2 + \frac{t_2 - t_3}{2} \right\| \geqslant \|t_1 - t_2\| - \frac{\|t_2 - t_3\|}{2} > \frac{\|t_2 - t_3\|}{2}. \qquad \square$$

Remark 3.1. It follows from Proposition 3.1 that if $\|t_1 - t_2\| > \|t_1 - t_3\| > \|t_2 - t_3\|$, then $T(t_1, t_2, t_3)$ is norm-acute at t_1 and at t_2. At the vertex t_3, the triangle can be of any of the three types (see Fig. 3(A, B, C)).

Proposition 3.2. *Let $T(t_1, t_2, t_3)$ be an isosceles triangle in $(\mathbb{R}^2, \|\cdot\|)$ with $\|t_1 - t_2\| = \|t_1 - t_3\|$.*

(i) *If $\|t_1 - t_2\| = \|t_1 - t_3\| > \|t_2 - t_3\|$, then $T(t_1, t_2, t_3)$ is norm-acute at t_1, and it is neither norm-obtuse at t_2 nor at t_3.*
(ii) *If $\|t_1 - t_2\| = \|t_1 - t_3\| < \|t_2 - t_3\|$, then $T(t_1, t_2, t_3)$ is norm-acute at t_2 and at t_3.*

Proof. (i) From Proposition 3.1 it follows that $T(t_1, t_2, t_3)$ is norm-acute at t_1. For $\{i, j\} = \{2, 3\}$ we have

$$\left\| t_i - \frac{t_1 + t_j}{2} \right\| = \left\| t_i - t_1 - \frac{t_j - t_1}{2} \right\| \geqslant \|t_i - t_1\| - \frac{\|t_j - t_1\|}{2} = \frac{\|t_j - t_1\|}{2},$$

from which it follows that $T(t_1, t_2, t_3)$ is norm-obtuse neither at t_2 nor at t_3.

(ii) This follows from Proposition 3.1. $\square$

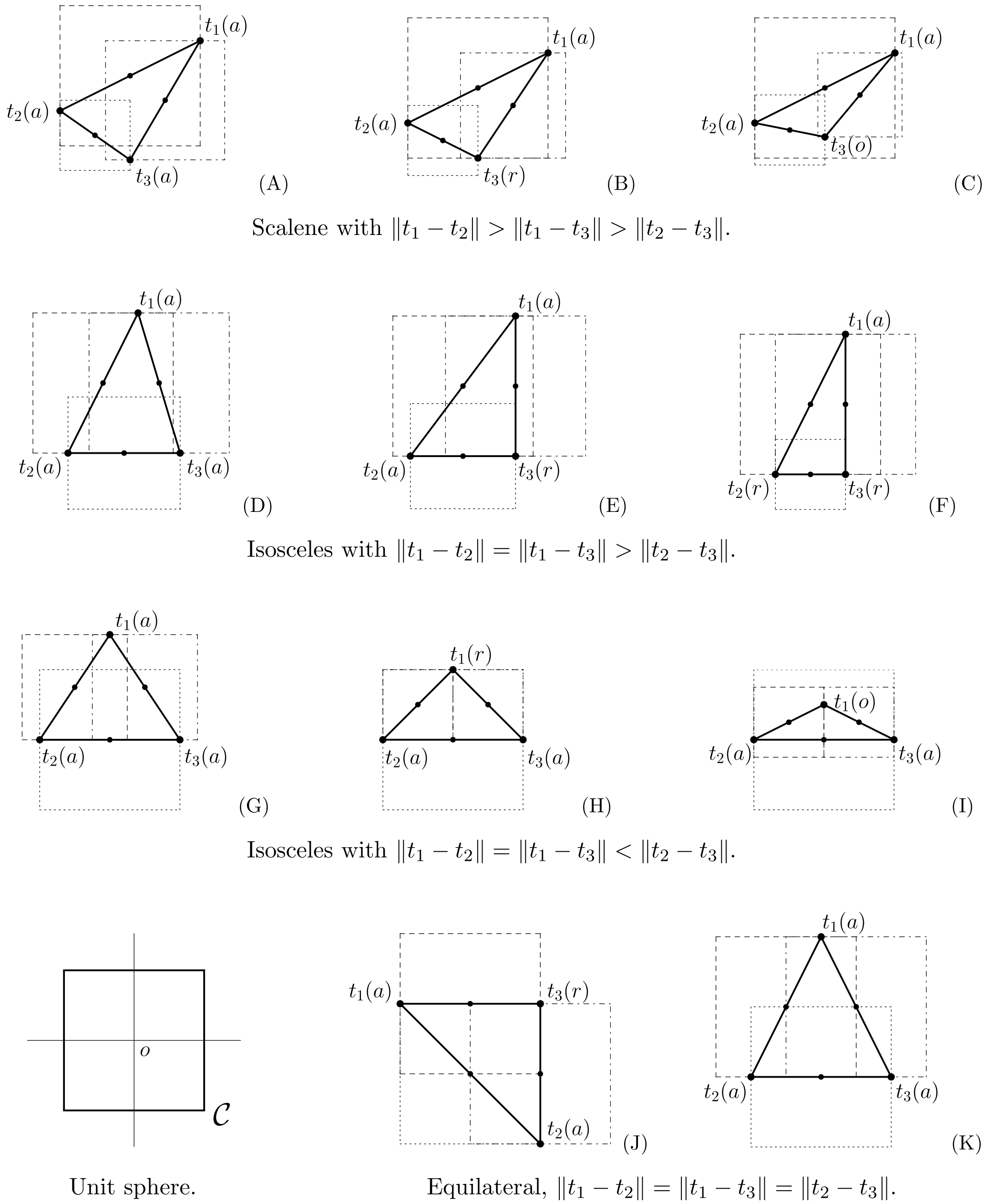


Fig. 3. Examples of types of triangles and vertices: (a) acute, (r) right, (o) obtuse.

Remark 3.2. In case (i) of Proposition 3.2, $T(t_1, t_2, t_3)$ can be either norm-acute or norm-right at t_2 and at t_3 (see Fig. 3(D, E, F)). And in case (ii), the triangle can be of any of the three types at the vertex t_1 (see Fig. 3(G, H, I)).

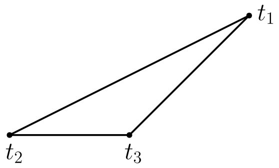
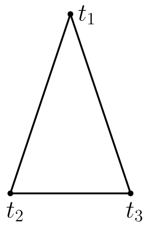
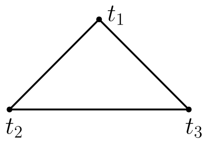
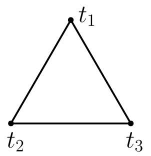
Proposition 3.3. *Let $T(t_1, t_2, t_3)$ be an equilateral triangle in $(\mathbb{R}^2, \|\cdot\|)$, i.e., $\|t_1 - t_2\| = \|t_1 - t_3\| = \|t_2 - t_3\|$. Then:*

(i) *$T(t_1, t_2, t_3)$ is norm-obtuse at no vertex.*
(ii) *$T(t_1, t_2, t_3)$ is norm-right at a vertex, say t_3, if and only if the unit disc is a parallelogram and the sides of this parallelogram are parallel to $[t_1, t_3]$ and $[t_2, t_3]$.*
(iii) *If $T(t_1, t_2, t_3)$ is norm-right at a vertex, then it is norm-acute at the other two vertices.*

Proof. (i) This follows as in case (i) of Proposition 3.2.

(ii) Assume that $T(t_1, t_2, t_3)$ is norm-right at t_3. Without loss of generality, we can assume that $\|t_1 - t_2\| = 2$ and that the midpoint of $[t_1, t_2]$ coincides with the origin. Then t_1, t_2 and t_3 lie on the unit circle $\mathcal{C}$. Thus the monotonicity lemma

Table 1
All the possible types of triangles and vertices.

Triangle type	t_1	t_2	t_3	Examples (Fig. 3)
 $\|t_1 - t_2\| > \|t_1 - t_3\| > \|t_2 - t_3\|$	acute	acute	acute	A
			right	B
			obtuse	C
 $\|t_1 - t_2\| = \|t_1 - t_3\| > \|t_2 - t_3\|$	acute	acute	acute	D
				E
		right	right	F
 $\|t_1 - t_2\| = \|t_1 - t_3\| < \|t_2 - t_3\|$	acute	acute	acute	G
	right			H
	obtuse			I
 $\|t_1 - t_2\| = \|t_1 - t_3\| = \|t_2 - t_3\|$	acute	acute	right	J
	acute	right	acute	
	right	acute	acute	
	acute	acute	acute	K

implies that the segments $[t_1, t_3]$ and $[t_2, t_3]$ belong to $\mathcal{C}$. Since $\|t_3 - t_2\| = \|t_3 - t_1\| = 2$, this is only possible when $\mathcal{C}$ is a parallelogram. The converse statement is trivial.

(iii) Assume that $T(t_1, t_2, t_3)$ is norm-right at t_3. It follows from (ii) that for $\{i, j\} = \{1, 2\}$

$$\left\| t_i - \frac{t_j + t_3}{2} \right\| = \|t_j - t_3\| > \frac{\|t_j - t_3\|}{2},$$

i.e., $T(t_1, t_2, t_3)$ is norm-acute at t_i for $i = 1, 2$. $\square$

Remark 3.3. It is obvious that an equilateral triangle can also be norm-acute at all three vertices (Fig. 3(K)).

The above results, summarized in Table 1, make the following definitions consistent.

Definition 3.1. We say that a triangle $T(t_1, t_2, t_3)$ in $(\mathbb{R}^2, \|\cdot\|)$ is

(i) norm-obtuse, if it is norm-obtuse at some vertex;
(ii) norm-right, if it is norm-right at exactly one vertex;
(iii) doubly norm-right, if it is norm-right at two vertices;
(iv) norm-acute, if it is norm-acute at the three vertices.

4. Minimal enclosing discs of norm-obtuse and norm-right triangles

In this section we deal with norm-obtuse and norm-right triangles. Note first that if $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at the vertex t_3, then $\|t_1 - t_2\| \geqslant \max\{\|t_1 - t_3\|, \|t_2 - t_3\|\}$ (see Table 1).

Proposition 4.1. *Assume that the triangle $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at the vertex t_3. Let $\lambda = \frac{1}{2}\|t_1 - t_2\|$. Then:*

(i) *$\mathcal{D}(\frac{1}{2}(t_1 + t_2), \lambda)$ is a minimal enclosing disc of t_1, t_2, t_3.*
(ii) *If $\mathcal{D}(c, \lambda)$ is another minimal enclosing disc of t_1, t_2, t_3, then $t_1, t_2 \in \mathcal{C}(c, \lambda)$.*
(iii) *$MEDC(t_1, t_2, t_3) = \mathcal{C}(t_1, \lambda) \cap \mathcal{C}(t_2, \lambda) \cap \mathcal{D}(t_3, \lambda)$.*

Proof. (i) This follows by recalling that $\|t_1 - t_2\| \geqslant \max\{\|t_1 - t_3\|, \|t_2 - t_3\|\}$, and that any disc containing t_1 and t_2 has radius greater than or equal to $\frac{1}{2}\|t_1 - t_2\|$.

(ii) It is enough to observe that $\frac{1}{2}\|t_1 - t_2\| \geqslant \|t_1 - c\| \geqslant \|t_1 - t_2\| - \|t_2 - c\| \geqslant \|t_1 - t_2\| - \frac{1}{2}\|t_1 - t_2\| = \frac{1}{2}\|t_1 - t_2\|$, from which it follows that $t_1 \in \mathcal{C}(c, \lambda)$. Similarly, we obtain also $t_2 \in \mathcal{C}(c, \lambda)$.

(iii) This follows directly from (i) and (ii). □

Under the hypothesis of Proposition 4.1, we say that the disc $D = \mathcal{D}(\frac{1}{2}(t_1 + t_2), \frac{1}{2}\|t_1 - t_2\|)$ is the *main minimal enclosing disc* of t_1, t_2, t_3, and we will refer to it as the *MME*-disc. In what follows our aim is to describe in detail and by different ways the set $MEDC(t_1, t_2, t_3)$. Let us begin by defining the set

$$B_d(t_1, t_2) := \mathcal{C}\left(t_1, \frac{1}{2}\|t_1 - t_2\|\right) \cap \mathcal{C}\left(t_2, \frac{1}{2}\|t_1 - t_2\|\right),$$

that we call the *d-bisector* of t_1 and t_2. Later we will justify that name.

The next theorem gives a characterization of the set $MEDC(t_1, t_2, t_3)$ based on $B_d(t_1, t_3)$ and the translation

$$\theta(x) = x + \frac{1}{2}(t_1 + t_2) - t_3.$$

Theorem 4.1. *Assume that $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at the vertex t_3, and that D is the main minimal enclosing disc of t_1, t_2, t_3. Then*

$$MEDC(t_1, t_2, t_3) = \left\{x \in \mathbb{R}^2\colon\ \theta(x) \in \theta\big(B_d(t_1, t_2)\big) \cap D\right\}.$$

Proof. Let $\lambda = \frac{1}{2}\|t_1 - t_2\|$, and assume that $\mathcal{D}(x, \lambda)$ is a minimal enclosing disc of t_1, t_2, t_3. By Proposition 4.1(ii) we know that $x \in B_b(t_1, t_2)$. Moreover,

$$\left\|\theta(x) - \frac{1}{2}(t_1 + t_2)\right\| = \|x - t_3\| \leqslant \lambda, \tag{1}$$

and then $\theta(x) \in D$. Conversely, assume that $\theta(x) \in \theta(B_d(t_1, t_2)) \cap D$. Then $x \in B_b(t_1, t_2) = \mathcal{C}(t_1, \lambda) \cap \mathcal{C}(t_2, \lambda)$, and since $\theta(x) \in D$, it follows from (1) that $x \in \mathcal{D}(t_3, \lambda)$. Proposition 4.1(iii) then implies that $x \in MEDC(t_1, t_2, t_3)$. □

Let us justify now why we call $B_d(t_1, t_2)$ the d-bisector of t_1 and t_2. For $x, y \in (\mathbb{R}^2, \|\cdot\|)$, the sets

$$[x, y]_d := \left\{z \in \mathbb{R}^2\colon\ \|x - y\| = \|x - z\| + \|z - y\|\right\}$$

and

$$B(x, y) := \left\{z \in \mathbb{R}^2\colon\ \|x - z\| = \|y - z\|\right\}$$

are, respectively, called the *d-segment* and the *bisector* of x and y. It is straightforward to check that

$$B_d(x, y) = B(x, y) \cap [x, y]_d.$$

The shape of d-segments can be described in terms of the structure of the unit sphere $\mathcal{C}$; see, e.g., [6, §9] and [19, §3.2]. Namely, if x is an extreme point of $\mathcal{C}(y, \|x - y\|)$ (and therefore y is an extreme point of $\mathcal{C}(x, \|x - y\|)$), then $[x, y]_d = [x, y]$. On the contrary, if x is not an extreme point of $\mathcal{C}(y, \|x - y\|)$ (see Fig. 4), then let F_x be the maximal segment of $\mathcal{C}(x, \|x - y\|)$ having y in its relative interior. Since then y is not an extreme point of $\mathcal{C}(x, \|x - y\|)$, F_y is defined analogously. Note that F_x and F_y are symmetric with respect to $(x + y)/2$. Consider the cones $C_x = \{x + \lambda(u - x)\colon\ u \in F_x, \lambda \geqslant 0\}$ and $C_y = \{y + \lambda(v - y)\colon\ v \in F_y, \lambda \geqslant 0\}$. Then $[x, y]_d = C_x \cap C_y$, and therefore $[x, y]_d$ is a non-degenerate parallelogram. Let $M(x, y)$ be the line through the midpoint of $[x, y]$ and parallel to the segments F_x and F_y. (For more properties of this line we refer to [6, p. 56].) If x and y are extreme points, then we assume that this line degenerates to the midpoint of $[x, y]$.

The next proposition shows that $B_d(x, y)$ is always a segment, which can degenerate to a point.

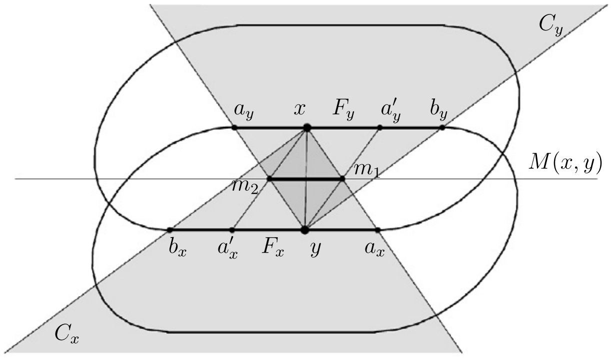


Fig. 4. The d-bisector $[x, y]_d$ when x is a non-extreme point of $\mathcal{C}(y, \|x-y\|)$.

Proposition 4.2. *Let there be given two different points x and y in $(\mathbb{R}^2, \|\cdot\|)$. Then*

$$B_d(x, y) = M(x, y) \cap [x, y]_d.$$

Proof. If $[x, y]_d = [x, y]$, then the statement is evidently true. Assume, on the contrary, that $[x, y]_d$ is a non-degenerate parallelogram, and let $F_x = [a_x, b_x]$ and $F_y = [a_y, b_y]$ be determined as above, with a_x, a_y symmetric with respect to the midpoint of $[x, y]$ (see Fig. 4). Moreover, assume that $\|a_x - y\| \leqslant \|b_x - y\|$. Let a'_x be the point symmetric to a_x with respect to y, i.e., $a'_x = 2y - a_x$, and similarly, let $a'_y = 2x - a_y$. Then

$$M(x, y) \cap [x, y]_d = [m_1, m_2],$$

where $m_1 = \langle x, a_x\rangle \cap \langle y, a'_y\rangle$ and $m_2 = \langle x, a'_x\rangle \cap \langle y, a_y\rangle$. Moreover, if $z \in [m_1, m_2]$, then $z \in B(x, y)$, because $\|x - z\| = \frac{1}{2}\|x - y\| = \|y - z\|$. Thus, we get $M(x, y) \cap [x, y]_d \subset B(x, y) \cap [x, y]_d$. For the converse, take $u \in B(x, y) \cap [x, y]_d$. Then $\|x - y\| = \|x - u\| + \|u - y\| = 2\|x - u\|$, i.e., u is the midpoint of $[x, u']$, where u' is the intersection point of the ray $[x, u\rangle$ and F_x, which implies that $u \in M(x, y)$. $\square$

It is interesting to note that (as we see in the proof of the above proposition) $B_d(x, y)$ is determined by the extreme of F_y closest to x, i.e., by a_y, whereas the other extreme, b_y, plays no role in the shape of $B_d(x, y)$.

Once we know the geometric shape of d-bisectors, let us return to the study of the minimal enclosing discs of a norm-obtuse or norm-right triangle.

Proposition 4.3. *If $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at t_3, and t_1 (equivalently, t_2) is an extreme point of the MME-disc, then the only minimal enclosing disc of t_1, t_2, t_3 is the MME-disc.*

Proof. It is enough to consider that if t_1 is an extreme point, then $B_d(t_1, t_2) = \frac{1}{2}(t_1 + t_2)$. $\square$

Assume now that $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at the vertex t_3, but t_1 is not an extreme point of the *MME*-disc D. In that case the d-bisector of t_1 and t_2 is a non-degenerate segment, i.e., $B_d(t_1, t_2) = [m_1, m_2]$, with $m_1 \neq m_2$. Since $t_3 \in D$, then there exists $\lambda \in \mathbb{R}$ such that $t_3 + \lambda(m_1 - m_2) \in \mathcal{C}(\frac{1}{2}(t_1 + t_2), \frac{1}{2}\|t_1 - t_2\|)$. Let λ_- and λ_+ be the smallest and the largest, respectively, of such λ's. Then $\lambda_- \leqslant 0 \leqslant \lambda_+$. Moreover, since $t_1 \neq t_3 \neq t_2$, we have that $\lambda_- < \lambda_+$. The next theorem describes the set $MEDC(t_1, t_2, t_3)$ by means of the above parameters.

Theorem 4.2. *Assume that $T(t_1, t_2, t_3)$ is norm-obtuse or norm-right at the vertex t_3 and that t_1 is not an extreme point of the MME-disc. Then*

$$MEDC(t_1, t_2, t_3) = \left\{\mu m_1 + (1-\mu)m_2\colon \max\left\{0, \frac{1}{2} - \lambda_+\right\} \leqslant \mu \leqslant \min\left\{1, \frac{1}{2} - \lambda_-\right\}\right\}.$$

Proof. Assume that $\mathcal{D}(x, \frac{1}{2}\|t_1 - t_2\|)$ is a minimal enclosing disc of t_1, t_2, t_3. By Proposition 4.1 we know that $x \in [m_1, m_2]$, which implies that $x = \mu m_1 + (1-\mu)m_2$ with $0 \leqslant \mu \leqslant 1$. Moreover, since $t_1 + t_2 = m_1 + m_2$, we have that

$$\frac{1}{2}\|t_1 - t_2\| \geqslant \|t_3 - x\| = \left\|t_3 - \frac{1}{2}(t_1 + t_2) + \left(\frac{1}{2} - \mu\right)(m_1 - m_2)\right\| = h\left(\frac{1}{2} - \mu\right),$$

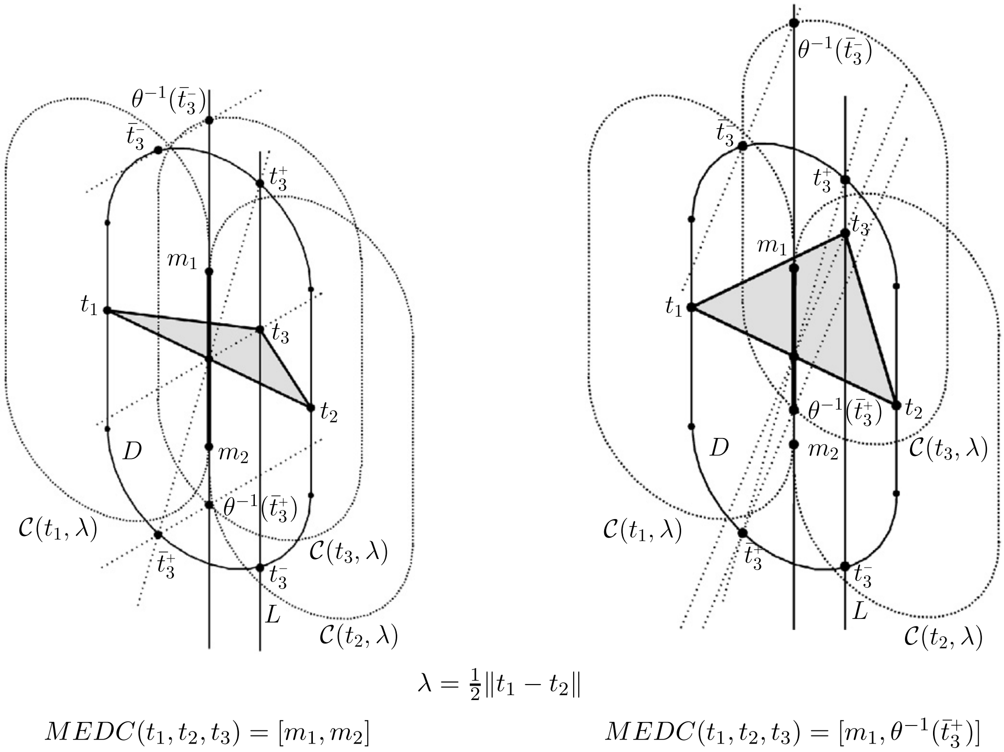


Fig. 5. Locus of the centers of the minimal enclosing discs of a norm-obtuse triangle.

where

$$h(\lambda) := \left\| t_3 - \frac{1}{2}(t_1 + t_2) + \lambda(m_1 - m_2) \right\|$$

is a convex function that satisfies $h(\lambda_-) = h(\lambda_+) = \frac{1}{2}\|t_1 - t_2\|$, λ_- and λ_+ being, respectively, the smallest and the largest λ that satisfy the last equality. This implies that $\lambda_- \leqslant \frac{1}{2} - \mu \leqslant \lambda_+$, from which it follows that

$$\max\left\{0, \frac{1}{2} - \lambda_+\right\} \leqslant \mu \leqslant \min\left\{1, \frac{1}{2} - \lambda_-\right\}. \tag{2}$$

Conversely, assume that (2) holds. Then $x \in B_b(t_1, t_2)$, which implies that $t_1, t_2 \in \mathcal{D}(x, \frac{1}{2}\|t_1 - t_2\|)$. Let $\gamma = \frac{2(\mu+\lambda_+)-1}{2(\lambda_+-\lambda_-)}$. Then $0 \leqslant \gamma \leqslant 1$. Since $m_1 + m_2 = t_1 + t_2$ and $t_3 + \lambda_\pm(m_1 - m_2) \in D$, we have that

$$\begin{aligned}
\|x - t_3\| &= \left\| \mu m_1 + (1-\mu) m_2 - t_3 \right\| \\
&= \left\| \gamma\left(\frac{1}{2}(t_1 + t_2) - \big(t_3 + \lambda_-(m_1 - m_2)\big)\right) + (1-\gamma)\left(\frac{1}{2}(t_1 + t_2) - \big(t_3 + \lambda_+(m_1 - m_2)\big)\right)\right\| \\
&\leqslant \gamma\left\| \frac{1}{2}(t_1 + t_2) - \big(t_3 + \lambda_-(m_1 - m_2)\big)\right\| + (1-\gamma)\left\| \frac{1}{2}(t_1 + t_2) - \big(t_3 + \lambda_+(m_1 - m_2)\big)\right\| \\
&= \frac{1}{2}\|t_1 - t_2\|,
\end{aligned}$$

which completes the proof. □

Remark 4.1. Assume that t_1 is not an extreme point, and let $t_3^+ = t_3 + \lambda_+(m_1 - m_2)$ and $t_3^- = t_3 + \lambda_-(m_1 - m_2)$; see Fig. 5. Then $[t_3^+, t_3^-] = D \cap L$, where L is the line through t_3 parallel to $\langle m_1, m_2 \rangle$. Let $\bar{t}_3^+$ and $\bar{t}_3^-$ be, respectively, the points symmetric to t_3^+ and t_3^- with respect to $\frac{1}{2}(t_1 + t_2)$, i.e.,

$$\bar{t}_3^+ = t_1 + t_2 - t_3 - \lambda_+(m_1 - m_2), \qquad \bar{t}_3^- = t_1 + t_2 - t_3 - \lambda_-(m_1 - m_2).$$

(a) The condition $\theta(x) \in \theta(B_d(t_1, t_2)) \cap D$ in Theorem 4.1 is equivalent to

$$x \in [m_1, m_2] \cap \left[\theta^{-1}\left(\bar{t}_3^-\right), \theta^{-1}\left(\bar{t}_3^+\right)\right], \tag{3}$$

where $\theta^{-1}(\bar{t}_3^{\pm}) = \frac{1}{2}(t_1 + t_2) - \lambda_{\pm}(m_1 - m_2)$.

(b) Having in mind that the four points $t_i \pm \frac{1}{2}(m_1 - m_2)$, $i = 1, 2$, belong to D, it is easy to prove (with the help of the function $h(\lambda)$ of Theorem 4.2) that $\lambda_+ - \lambda_- \geqslant 1$. This makes it possible that $[0, 1] \subseteq [\frac{1}{2} - \lambda_+, \frac{1}{2} - \lambda_-]$. If t_3 is such that this inclusion occurs, then from Theorem 4.2 it follows that $MEDC(t_1, t_2, t_3) = B_d(t_1, t_2)$, and therefore any minimal enclosing disc of t_1, t_2 also encloses t_3.

(c) Note that $[0, 1] \subseteq [\frac{1}{2} - \lambda_+, \frac{1}{2} - \lambda_-]$ is equivalent to $[m_1, m_2] \subseteq [\theta^{-1}(\bar{t}_3^-), \theta^{-1}(\bar{t}_3^+)]$. Therefore, if the last inclusion holds, the same conclusion as in (b) follows. Note also that if this inclusion does not hold, then $[m_1, m_2] \cap [\theta^{-1}(\bar{t}_3^-), \theta^{-1}(\bar{t}_3^+)]$ is either $[m_1, \theta^{-1}(\bar{t}_3^+)]$ or $[\theta^{-1}(\bar{t}_3^-), m_2]$.

Next we will see that if $T(t_1, t_2, t_3)$ is a norm-right or norm-obtuse triangle, then the set $MEDC(t_1, t_2, t_3)$ is the Fermat–Torricelli locus of certain points.

Recall that a point x_0 is called a *Fermat–Torricelli point* of given points $x_1, \ldots, x_n$ if $x = x_0$ minimizes the function $\sum_{i=1}^{n} \|x - x_i\|$. The *Fermat–Torricelli locus* $FT(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n$ is the set of all the Fermat–Torricelli points of $x_1, \ldots, x_n$. Note that since $FT(x_1, \ldots, x_n)$ is a convex set, it has a point not from $\{x_1, \ldots, x_n\}$ if it is not a singleton.

For a given normed space $(\mathbb{R}^d, \|\cdot\|)$, let us consider the dual space $(\mathbb{R}^d)^*$ endowed with the norm

$$\|\phi\| := \max_{\|x\|=1} \phi(x)$$

for any linear functional ϕ. A *norming functional* of $x \in (\mathbb{R}^d, \|\cdot\|)$ is a functional $\phi_x \in (\mathbb{R}^d)^*$ with $\|\phi_x\| = 1$ and $\phi_x(x) = \|x\|$. Note that if $\|x\| = 1$, the hyperplane $\phi_x^{-1}(1) = \{y \in \mathbb{R}^d\colon \phi_x(y) = 1\}$ is then a supporting hyperplane of the unit ball at x. By the Hahn–Banach theorem, each $x \in \mathbb{R}^d$ has a norming functional which is unique if and only if $(\mathbb{R}^d, \|\cdot\|)$ is smooth. Given a functional $\phi \in (\mathbb{R}^d)^*$ and a point $x \in \mathbb{R}^d$, define the cone

$$C(x, \phi) := x - \left\{a\colon \phi(a) = \|a\|\right\},$$

i.e., $C(x, \phi)$ is the translate by x of the union of all rays from the origin through the extreme face $\phi^{-1}(-1) \cap B$, where B is the unit ball of $(\mathbb{R}^d, \|\cdot\|)$. The following characterization of the Fermat–Torricelli locus is known; see [10] and [21, Theorem 3.2].

Theorem 4.3. *(See [10,21].) Let $x_1, \ldots, x_n$ be different points in $(\mathbb{R}^d, \|\cdot\|)$. Assume that there exists $p \in FT(x_1, \ldots, x_n) \setminus \{x_1, \ldots, x_n\}$, and that for each $i = 1, \ldots, n$ there exists a norming functional ϕ_i of $x_i - p$ such that $\sum_{i=1}^{n} \phi_i = 0$. Then*

$$FT(x_1, \ldots, x_n) = \bigcap_{i=1}^{n} C(x_i, \phi_i).$$

The above theorem, together with Theorem 4.1, leads to the following corollary, where $\bar{t}_3^+$ and $\bar{t}_3^-$ have been defined in Remark 4.1.

Corollary 4.1. *Let $T(t_1, t_2, t_3)$ be a triangle which is norm-right or norm-obtuse at the vertex t_3. Assume that t_1 is not an extreme point of D, and that the points of $C(\frac{1}{2}(t_1 + t_2), \frac{1}{2}\|t_1 - t_2\|) \cap \langle m_1, m_2 \rangle$ are extreme. Then*

$$MEDC(t_1, t_2, t_3) = FT\left(t_1, t_2, \theta^{-1}\left(\bar{t}_3^+\right), \theta^{-1}\left(\bar{t}_3^-\right)\right).$$

5. Minimal enclosing discs of doubly norm-right triangles

In this section we deal with doubly norm-right triangles, i.e., triangles that are norm-right at two vertices. Recall (see Table 1) that if $T(t_1, t_2, t_3)$ is norm-right at the vertices t_1 and t_2, then $\|t_3 - t_1\| = \|t_3 - t_2\| > \|t_1 - t_2\|$ and

$$\left\|t_1 - \frac{1}{2}(t_2 + t_3)\right\| = \frac{1}{2}\|t_2 - t_3\| = \frac{1}{2}\|t_1 - t_3\| = \left\|t_2 - \frac{1}{2}(t_1 + t_3)\right\|,$$

from which it follows that t_1, t_2, t_3 belong to the circles $C(\frac{1}{2}(t_2 + t_3), \lambda)$ and $C(\frac{1}{2}(t_1 + t_3), \lambda)$, where $\lambda = \frac{1}{2}\|t_1 - t_3\|$. Therefore, any minimal enclosing disc that contains t_1, t_2, t_3 must have radius λ.

Lemma 5.1. *Let $T(t_1, t_2, t_3)$ be a triangle norm-right at t_1 and at t_2. Let $m_{13} = \frac{1}{2}(t_1 + t_3)$, $m_{23} = \frac{1}{2}(t_2 + t_3)$ and $\lambda = \frac{1}{2}\|t_1 - t_3\|$.*

(i) *If $\mathcal{D}(x, \lambda)$ is a minimal enclosing disc of t_1, t_2, t_3, then $t_1, t_2, t_3 \in \mathcal{C}(x, \lambda)$.*
(ii) *$C(\frac{1}{2}(m_{13} + m_{23}), \lambda)$ contains the line segment $[t_1 + \frac{1}{2}(m_{13} - m_{23}), t_2 + \frac{1}{2}(m_{23} - m_{13})]$.*
(iii) *If $\mathcal{D}(x, \lambda)$ is a minimal enclosing disc of t_1, t_2, t_3, then $[t_1, t_2] \subset \mathcal{C}(x, \lambda)$.*
(iv) *$[m_{13}, m_{23}] \subset MEDC(t_1, t_2, t_3) \subset \langle m_{13}, m_{23}\rangle$.*

Proof. (i) Assume that $t_1, t_2, t_3 \in \mathcal{D}(x, \lambda)$. If for $i = 1, 2$ a point of the couple $\{t_i, t_3\}$ is in the interior of $\mathcal{D}(x, \lambda)$, then $\|t_i - t_3\| < 2\lambda$, against the hypothesis.

(ii) Consider the convex function

$$\begin{aligned} f(\alpha) &= \left\|\alpha(m_{23} - t_1) + (1 - \alpha)(m_{13} - t_2)\right\| \\ &= \left\|\frac{1}{2}(m_{13} + m_{23}) - \alpha\left(t_1 + \frac{1}{2}(m_{13} - m_{23})\right) - (1 - \alpha)\left(t_2 + \frac{1}{2}(m_{23} - m_{13})\right)\right\|. \end{aligned}$$

Then $f(0) = \|m_{13} - t_2\| = \lambda$, $f(1/3) = \|m_{23} - t_2\| = \lambda$, $f(2/3) = \|m_{13} - t_1\| = \lambda$ and $f(1) = \|m_{23} - t_1\| = \lambda$, from which it follows that $f(\alpha) = \lambda$ for $0 \leqslant \alpha \leqslant 1$.

(iii) It follows from (i), (ii) and [1, Corollary 3.1].

(iv) Let $0 \leqslant \beta \leqslant 1$. Taking $\alpha_1 = \frac{3-\beta}{3}$, $\alpha_2 = \frac{1-\beta}{3}$ and $\alpha_3 = \frac{1+\beta}{3}$, we have that $0 \leqslant \alpha_i \leqslant 1$, $i = 1, 2, 3$, and

$$\begin{aligned} \beta m_{13} + (1 - \beta)m_{23} - t_1 &= \alpha_1(m_{23} - t_1) + (1 - \alpha_1)(m_{13} - t_2), \\ \beta m_{13} + (1 - \beta)m_{23} - t_2 &= \alpha_2(m_{23} - t_1) + (1 - \alpha_2)(m_{13} - t_2), \\ t_3 - \beta m_{13} - (1 - \beta)m_{23} &= \alpha_3(m_{23} - t_1) + (1 - \alpha_3)(m_{13} - t_2). \end{aligned}$$

Since $f(\alpha_i) = \lambda$, $i = 1, 2, 3$, we get that $[m_{13}, m_{23}] \subset MEDC(t_1, t_2, t_3)$.

Let $p = \frac{1}{2}(m_{13} + m_{23})$. Then $\mathcal{D}(p, \lambda)$ is a minimal enclosing disc of t_1, t_2, t_3. Let $\mathcal{D}(x, \lambda)$ be another minimal enclosing disc of t_1, t_2, t_3. By (iii) we know that $[t_1, t_2] \subset \mathcal{C}(p, \lambda) \cap \mathcal{C}(x, \lambda)$, and from [1, Theorem 4.2, (b)] it follows that $x \in \langle m_{13}, m_{23}\rangle \cup K_1 \cup K_2$, where

$$K_i = \left\{t_i + \mu_i(t_{3-i} - t_i) + \frac{1}{2}(t_3 - t_i)\colon \frac{3}{4} \leqslant \mu_i\right\}, \quad i = 1, 2.$$

Assume that $x \in K_1$, i.e., $x = t_1 + \mu_1(t_2 - t_1) + \frac{1}{2}(t_3 - t_1)$. Taking $\alpha = 2\mu_1$, we get $x = (1 - \alpha)m_{13} + \alpha m_{23}$, from which it follows that $K_1 \subset \langle m_{13}, m_{23}\rangle$. Similarly, $K_2 \subset \langle m_{13}, m_{23}\rangle$. □

From the above lemma we know that the disc $\mathcal{D}(p, \lambda)$, with $p = \frac{1}{2}(m_{13} + m_{23})$ and $\lambda = \frac{1}{2}\|t_1 - t_3\| = \frac{1}{2}\|t_2 - t_3\|$, is a minimal enclosing disc for t_1, t_2, t_3. In the present situation, we call this disc the *main minimal enclosing disc* (*MME*-disc) of t_1, t_2, t_3.

Theorem 5.1. *Let $T(t_1, t_2, t_3)$ be a triangle norm-right at t_1 and at t_2, with MME-disc $\mathcal{D}(p, \lambda)$. Let $[u_1, u_2]$ be the longest line segment of $\mathcal{C}(p, \lambda)$ containing $[t_1, t_2]$ such that u_1, t_1, t_2, u_2 are placed in this order. Let p_1 be that point of the pair $\{t_2 + p - u_2, t_3 + u_1 - p\}$, and p_2 that point of $\{t_1 + p - u_1, t_3 + u_2 - p\}$, which is closer to p, in each case (see Fig. 6). Then*

$$MEDC(t_1, t_2, t_3) = [p_1, p_2] = \mathcal{G}(t_1, \lambda) \cap \mathcal{G}(t_2, \lambda) \cap \mathcal{G}(t_3, \lambda),$$

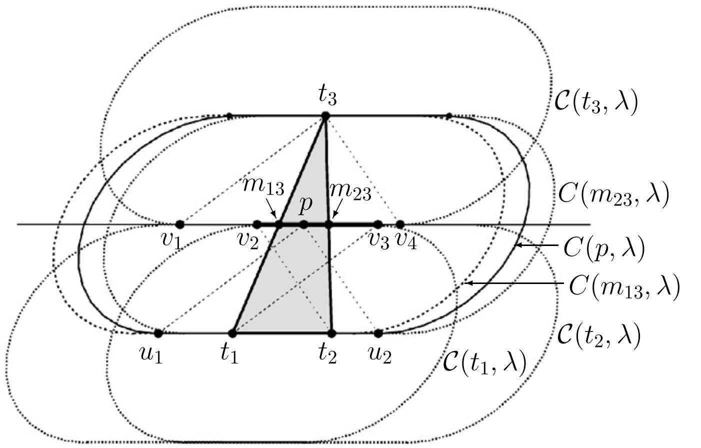
where "$\mathcal{G}$" means, indistinctly, "$\mathcal{C}$" or "$\mathcal{D}$". Moreover, if $u_1 = t_1 + \mu_1(t_1 - t_2)$ and $u_2 = t_2 + \mu_2(t_2 - t_1)$, with $\mu_1, \mu_2 \geqslant 0$, then $x = (1 - \beta)m_{13} + \beta m_{23}$ belongs to $[p_1, p_2]$ if and only if $\beta \in [\beta_1, \beta_2]$, where

$$[\beta_1, \beta_2] = \begin{cases} [\frac{1}{2} - 2\mu_2, \frac{3}{2} + 2\mu_2] & \text{if } \mu_2 \leqslant \mu_1 - \frac{1}{2}, \\ [\frac{1}{2} - 2\mu_2, \frac{1}{2} + 2\mu_1] & \text{if } \mu_1 - \frac{1}{2} \leqslant \mu_2 \leqslant \mu_1 + \frac{1}{2}, \\ [-\frac{1}{2} - 2\mu_1, \frac{1}{2} + 2\mu_1] & \text{if } \mu_2 \geqslant \mu_1 + \frac{1}{2}. \end{cases} \tag{4}$$

Proof. First observe that, since $t_1 + \frac{1}{2}(m_{13} - m_{23}) = t_1 + \frac{1}{4}(t_1 - t_2)$ and $t_2 + \frac{1}{2}(m_{23} - m_{13}) = t_2 + \frac{1}{4}(t_2 - t_1)$, it follows from Lemma 5.1(ii) that $\mu_i \geqslant \frac{1}{4}$, $i = 1, 2$. Consider the convex function

$$f(\mu) = \left\|p - (1 - \mu)t_1 - \mu t_2\right\|, \quad \lambda \in \mathbb{R}.$$

Then $f(-\mu_1) = \|p - (1 + \mu_1)t_1 + \mu_1 t_2\| = \|p - u_1\| = \lambda$ and $f(1 + \mu_2) = \|p + \mu_2 t_1 - (1 + \mu_2)t_2\| = \|p - u_2\| = \lambda$. Since $[u_1, u_2]$ is the longest segment of $\mathcal{C}(p, \lambda)$ that contains t_1 and t_2, it follows that $f(\mu) = \lambda$ for every $\lambda \in [-\mu_1, 1 + \mu_2]$. Moreover, for μ out of this segment, $f(\mu) > \lambda$.



$$v_1 = t_3 + u_1 - p, \quad v_2 = t_2 + p - u_2,$$
$$v_3 = t_1 + p - u_1, \quad v_4 = t_3 + u_2 - p, \qquad \lambda = \tfrac{1}{2}\|t_1 - t_3\| = \tfrac{1}{2}\|t_2 - t_3\|$$
$$MEDC(t_1,t_2,t_3) = [p_1,p_2] = [v_2,v_3]$$

Fig. 6. Locus of the centers of the minimal enclosing discs of a doubly norm-right triangle.

Consider now the identities

$$t_2 + p - u_2 = (1-\gamma_1)m_{13} + \gamma_1 m_{23}, \qquad t_3 + u_1 - p = (1-\gamma_2)m_{13} + \gamma_2 m_{23},$$
$$t_1 + p - u_1 = (1-\gamma_3)m_{13} + \gamma_3 m_{23}, \qquad t_3 + u_2 - p = (1-\gamma_4)m_{13} + \gamma_4 m_{23},$$

where

$$\gamma_1 := \frac{1}{2} - 2\mu_2 \leqslant 0, \qquad \gamma_2 := -\frac{1}{2} - 2\mu_1 \leqslant 0, \qquad \gamma_3 := \frac{1}{2} + 2\mu_1 \geqslant 1, \qquad \gamma_4 := \frac{3}{2} + 2\mu_2 \geqslant 1.$$

Thus, to have $t_2 + p - u_2$ closer to p than $t_3 + u_1 - p$ is equivalent to $\gamma_1 \geqslant \gamma_2$, i.e., equivalent to $\mu_2 \leqslant \frac{1}{2} + \mu_1$. Similarly, to have $t_1 + p - u_1$ closer to p than $t_3 + u_2 - p$ is equivalent to $\gamma_3 \leqslant \gamma_4$, i.e., equivalent to $\mu_1 - \frac{1}{2} \leqslant \mu_2$. These imply that $x = (1-\beta)m_{13} + \beta m_{23}$ belongs to $[p_1, p_2]$ if and only if $\beta \in [\beta_1, \beta_2]$, where this interval is defined in (4).

Let $x = (1-\beta)m_{13} + \beta m_{23}$, $\beta \in \mathbb{R}$. Then the following identities hold:

$$x - t_1 = p - (1-\alpha_1)t_1 - \alpha_1 t_2,$$
$$x - t_2 = p - (1-\alpha_2)t_1 - \alpha_2 t_2,$$
$$t_3 - x = p - (1-\alpha_3)t_1 - \alpha_3 t_2,$$

where

$$\alpha_1 = \frac{1-2\beta}{4}, \qquad \alpha_2 = \frac{5-2\beta}{4}, \qquad \alpha_3 = \frac{1+2\beta}{4}.$$

Assume that $\beta \in [\beta_1, \beta_2]$. It is immediate to verify (in each of the three cases in (4)) that then $\alpha_1, \alpha_2, \alpha_3 \in [-\mu_1, 1+\mu_2]$, which implies that $f(\alpha_i) = \lambda$, $i = 1,2,3$, i.e., $t_i \in \mathcal{C}(x,\lambda)$, $i = 1,2,3$. This proves that $[p_1, p_2] \subset MEDC(t_1,t_2,t_3)$.

Conversely, assume that $x \in MEDC(t_1,t_2,t_3)$. By Lemma 5.1(iv), $x = (1-\beta)m_{13} + \beta m_{23}$, $\beta \in \mathbb{R}$. From the above identities it follows that $f(\alpha_i) = \lambda$, $i = 1,2,3$, and then $\alpha_1, \alpha_2, \alpha_3 \in [-\mu_1, 1+\mu_2]$. Also now it is immediate to verify that this implies that $\beta \in [\frac{1}{2} - 2\mu_2, \frac{3}{2} + 2\mu_2] \cup [\frac{1}{2} - 2\mu_2, \frac{1}{2} + 2\mu_1] \cup [-\frac{1}{2} - 2\mu_1, \frac{1}{2} + 2\mu_1]$, and then $\beta \in [\beta_1, \beta_2]$. Thus we get $MEDC(t_1,t_2,t_3) \subset [p_1, p_2]$, which completes the proof.

Finally, the identity $MEDC(t_1,t_2,t_3) = \mathcal{G}(t_1,\lambda) \cap \mathcal{G}(t_2,\lambda) \cap \mathcal{G}(t_3,\lambda)$ follows directly from Lemma 5.1(i). □

6. Minimal enclosing discs of norm-acute triangles

Theorem 6.1. *Any norm-acute triangle in* $(\mathbb{R}^2, \|\cdot\|)$ *has at least one circumcircle.*

Proof. Let $T(t_1,t_2,t_3)$ be a norm-acute triangle with $\|t_1 - t_2\| \geqslant \max\{\|t_1 - t_3\|, \|t_2 - t_3\|\}$. To simplify the proof, we assume that $\frac{1}{2}(t_1 + t_2) = o$ and $\|t_1\| = 1$. Then, $t_1, t_2 \in \mathcal{C} = \mathcal{C}(o,1)$ and $t_3 \notin \mathcal{D} = \mathcal{D}(o,1)$. Let $y \in \mathcal{C} \cap HP^-_{t_3}(t_1,t_2)$ such that t_1 is Birkhoff orthogonal to y (see Fig. 7). Then $\mathcal{C} \subset H := HP^+_o(t_1, t_1 + y) \cap HP^+_o(t_2, t_2 + y)$. Moreover, since $\|t_1 - t_2\| \geqslant \max\{\|t_1 -$

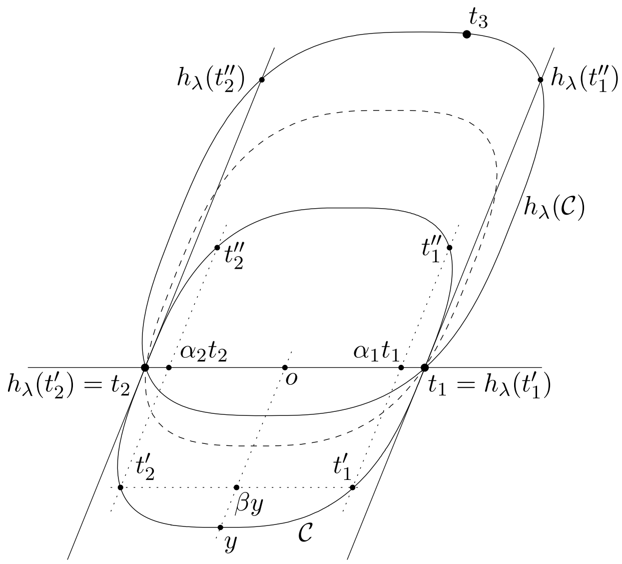


Fig. 7. Proof of Theorem 6.1.

$t_3\|, \|t_2 - t_3\|\}$, we have also $t_3 \in H$. We will see that we can inflate $\mathcal{C}$ such that the resulting circles rest on t_1 and t_2, until one of them catches t_3.

For any $0 < \lambda \leqslant 1$, let $t_1', t_2' \in \mathcal{C}$ be such that $t_1' - t_2' = \lambda(t_1 - t_2)$, and $t_1', t_2' \in HP^-_{t_3}(t_1, t_2)$. Then

$$t_1' = \alpha_1 t_1 + \beta y, \qquad t_2' = \alpha_2 t_1 + \beta y,$$

with $\beta > 0$ and $\alpha_1 - \alpha_2 = 2\lambda$. Since $t_1 \dashv y$, we have that $|\alpha_i| \leqslant 1$, $i = 1, 2$. Moreover, $\beta = \|t_1' - \alpha_1 t_1\| \leqslant 1 + |\alpha_1| \leqslant 2$, and there exists $\varepsilon_0 > 0$ such that $\beta > \varepsilon_0$ for λ small enough. Let $f(\mu) = \|\alpha_1 t_1 + \mu y\|$, $\mu \in \mathbb{R}$. Since $\lim_{\mu \to -\infty} f(\mu) = +\infty$ and $f(0) = |\alpha_1| \leqslant 1$, it follows that there exists $\beta_1 \leqslant 0$ such that $f(\beta_1) = 1$. Similarly with α_2. Thus there exist $t_1'', t_2'' \in \mathcal{C}$ with

$$t_1'' = \alpha_1 t_1 + \beta_1 y, \qquad t_2'' = \alpha_2 t_1 + \beta_2 y,$$

where $\beta_1 \leqslant 0$ and $\beta_2 \leqslant 0$. Moreover, we have also $|\beta_i| \leqslant 2$, $i = 1, 2$. Let us consider the homothety

$$h_\lambda(x) = \frac{1}{\lambda}\left(x - \frac{1}{2}\left(t_1' + t_2'\right)\right), \quad x \in \mathbb{R}.$$

Then

$$h_\lambda\left(t_1'\right) = t_1, \qquad h_\lambda\left(t_2'\right) = t_2, \qquad h_\lambda\left(t_1''\right) = t_1 + \left(\frac{\beta_1 - \beta}{\lambda}\right) y, \qquad h_\lambda\left(t_2''\right) = t_2 + \left(\frac{\beta_2 - \beta}{\lambda}\right) y.$$

Thus $t_1, t_2 \in h_\lambda(\mathcal{C})$. Since $\beta_i - \beta < 0$ and $\varepsilon_0 < |\beta_i - \beta| \leqslant 4$, $i = 1, 2$, we have that $\frac{\beta_i - \beta}{\lambda} \longrightarrow_{\lambda \to 0} -\infty$. If we move λ from 1 to 0, then the arc of $h_\lambda(\mathcal{C})$ between $h_\lambda(t_1'')$ and $h_\lambda(t_2'')$ that is in $H \cap HP^+_{t_3}(t_1, t_2)$ moves from an arc of $\mathcal{C}$ to arcs of circles that pass through t_1 and t_2, whose extremes $h_\lambda(t_1'')$ and $h_\lambda(t_2'')$ go away from t_1 and t_2, respectively, as much as we want. Necessarily, some of these circles give chase to t_3. $\square$

Next we will describe the set $MEDC(t_1, t_2, t_3)$ of any norm-acute triangle $T(t_1, t_2, t_3)$. To simplify the notation we consider

$$m_{12} = \frac{t_1 + t_2}{2}, \qquad m_{13} = \frac{t_1 + t_3}{2}, \qquad m_{23} = \frac{t_2 + t_3}{2}.$$

In [1] we defined the sets

$$K_0(t_1, t_2, t_3) := \operatorname{conv}\{m_{12}, m_{13}, m_{23}\},$$
$$K_i(t_1, t_2, t_3) := \left\{t_i + \alpha_1(t_j - t_i) + \alpha_2(t_k - t_i)\colon \alpha_1 \geqslant \frac{1}{2},\ \alpha_2 \geqslant \frac{1}{2}\right\}, \quad i = 1, 2, 3$$

(see Fig. 8), and we proved there the following theorem.

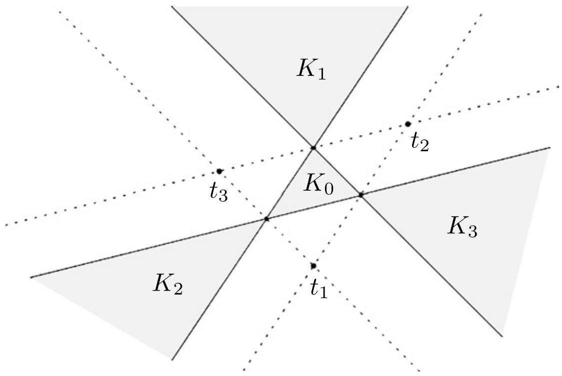


Fig. 8. The region where the center of a circle passing through t_1, t_2, t_3 has to be located.

Theorem 6.2. *(See [1].) Let t_1, t_2, t_3 be three non-collinear points in $\mathbb{R}^2$. There exist a norm $\|\cdot\|$ and a circle $\mathcal{C}(c, \lambda)$ in $(\mathbb{R}^2, \|\cdot\|)$ passing through the three points if and only if $c \in \bigcup_{i=0}^{3} K_i(t_1, t_2, t_3)$.*

Theorem 6.3. *Let t_1, t_2, t_3 be three non-collinear points in $(\mathbb{R}^2, \|\cdot\|)$ and assume that there exists a circle $\mathcal{C}(x, \lambda)$ passing through them.*

(i) *If $T(t_1, t_2, t_3)$ is norm-acute, then $x \in K_0(t_1, t_2, t_3) \setminus \{m_{12}, m_{13}, m_{23}\}$.*
(ii) *If x is an interior point of $K_0(t_1, t_2, t_3)$, then $T(t_1, t_2, t_3)$ is norm-acute, and $\mathcal{C}(x, \lambda)$ is the only circumcircle of $T(t_1, t_2, t_3)$.*

Proof. (i) Assume that $x \notin K_0(t_1, t_2, t_3) \setminus \{m_{12}, m_{13}, m_{23}\}$. Then we can assume, without loss of generality, that $x \in K_1(t_1, t_2, t_3)$, i.e., $x = t_1 + \alpha_1(t_2 - t_1) + \alpha_2(t_3 - t_1)$, with $\alpha_1 \geqslant \frac{1}{2}$, $\alpha_2 \geqslant \frac{1}{2}$. We can also assume that $\|m_{23} - t_3\| = 1$. Then $1 = \|t_3 - m_{23}\| = \|\frac{1}{2}(t_3 - x) + \frac{1}{2}(x - t_2)\| \leqslant \lambda$. Moreover,

$$\begin{aligned}(\alpha_1 + \alpha_2)\lambda &= \big\|(\alpha_1 + \alpha_2)(t_3 - x)\big\| = \big\|(\alpha_1 + \alpha_2 - 1)(t_1 - x) + 2\alpha_1(t_3 - m_{23})\big\| \\ &\leqslant (\alpha_1 + \alpha_2 - 1)\lambda + 2\alpha_1\end{aligned}$$

and

$$\begin{aligned}(\alpha_1 + \alpha_2)\lambda &= \big\|(\alpha_1 + \alpha_2)(t_2 - x)\big\| = \big\|(\alpha_1 + \alpha_2 - 1)(t_1 - x) + 2\alpha_2(t_2 - m_{23})\big\| \\ &\leqslant (\alpha_1 + \alpha_2 - 1)\lambda + 2\alpha_2,\end{aligned}$$

from which it follows that $\lambda \leqslant 2\alpha_1$ and $\lambda \leqslant 2\alpha_2$, respectively.

If $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$, then $x = m_{23}$, which implies that $T(t_1, t_2, t_3)$ is norm-right at t_1. Assume that $(\alpha_1, \alpha_2) \neq (\frac{1}{2}, \frac{1}{2})$. We will see that then $T(t_1, t_2, t_3)$ is norm-right or norm-obtuse at t_1. Let us consider the points $s_i = m_{23} + \frac{1}{\lambda}(t_i - x)$, $i = 1, 2, 3$. Then the following identities hold:

$$\begin{aligned}\gamma_1(t_1 - m_{23}) &= (1 - \eta_1)(s_2 - m_{23}) + \eta_1(t_2 - m_{23}), \\ \gamma_2(t_1 - m_{23}) &= (1 - \eta_2)(s_1 - m_{23}) + \eta_2(s_2 - m_{23}), \\ \gamma_3(t_1 - m_{23}) &= (1 - \eta_3)(s_3 - m_{23}) + \eta_3(s_1 - m_{23}), \\ \gamma_4(t_1 - m_{23}) &= (1 - \eta_4)(t_3 - m_{23}) + \eta_4(s_3 - m_{23}),\end{aligned} \tag{5}$$

where

$$\begin{aligned}&\gamma_1 = \frac{\alpha_1 + \alpha_2 - 1}{\alpha_1 - \alpha_2 - 1 + \lambda}, \qquad \gamma_2 = \frac{2\alpha_2}{\lambda}, \qquad \gamma_3 = \frac{2\alpha_1}{\lambda}, \qquad \gamma_4 = \frac{\alpha_1 + \alpha_2 - 1}{\alpha_2 - \alpha_1 - 1 + \lambda}, \\ &\eta_1 = \frac{\alpha_1 - \alpha_2 - 1}{\alpha_1 - \alpha_2 - 1 + \lambda}, \qquad \eta_2 = \alpha_1 - \alpha_2, \qquad \eta_3 = 1 + \alpha_1 - \alpha_2, \qquad \eta_4 = \frac{\lambda}{\alpha_2 - \alpha_1 - 1 + \lambda}.\end{aligned}$$

Let

$$\begin{aligned}U_1 &= \left\{(\nu_1, \nu_2)\colon\ \nu_1 \geqslant \frac{1}{2},\ \nu_2 \geqslant \frac{1}{2},\ \nu_2 \leqslant \nu_1 - 1\right\}, \\ U_2 &= \left\{(\nu_1, \nu_2)\colon\ \nu_1 \geqslant \frac{1}{2},\ \nu_2 \geqslant \frac{1}{2},\ \nu_1 - 1 \leqslant \nu_2 \leqslant \nu_1\right\},\end{aligned}$$

$$U_3 = \left\{(\nu_1, \nu_2)\colon\ \nu_1 \geqslant \frac{1}{2},\ \nu_2 \geqslant \frac{1}{2},\ \nu_1 \leqslant \nu_2 \leqslant \nu_1 + 1\right\},$$
$$U_4 = \left\{(\nu_1, \nu_2)\colon\ \nu_1 \geqslant \frac{1}{2},\ \nu_2 \geqslant \frac{1}{2},\ \nu_1 + 1 \leqslant \nu_2\right\}.$$

Then $(\alpha_1, \alpha_2) \in U_i$ for some i, which implies that $0 \leqslant \eta_i \leqslant 1$. Since $\lambda \leqslant 2\alpha_1$ and $\lambda \leqslant 2\alpha_2$, we have $\gamma_i \geqslant 1$. Moreover, from (5) it follows that $\gamma_i\|t_1 - m_{23}\| \leqslant 1$, which implies $\|t_1 - m_{23}\| \leqslant 1$, i.e., $T(t_1, t_2, t_3)$ is norm-right or norm-obtuse at t_1.

(ii) Assume that x is an interior point of K_0. Then there exist $\alpha_1, \alpha_2 \in (0, \frac{1}{2})$, $\alpha_1 + \alpha_2 < \frac{1}{2}$, such that $x = m_{23} + \alpha_1(t_1 - t_2) + \alpha_2(t_1 - t_3)$. As in the above case, we can assume that $\|m_{23} - t_3\| = 1$, and then $\lambda \geqslant 1$. Assume that $\alpha_1 \geqslant \alpha_2$. Then

$$\begin{aligned}(1 - 2\alpha_2)\|t_1 - m_{23}\| &= \big\|(1 + \alpha_1 - \alpha_2)(t_1 - x) + (\alpha_1 - \alpha_2)(x - t_2)\big\|\\ &\geqslant (1 + \alpha_1 - \alpha_2)\lambda - (\alpha_1 - \alpha_2)\lambda = \lambda \geqslant 1,\end{aligned}$$

from which it follows that $\|t_1 - m_{23}\| \geqslant 1/(1 - 2\alpha_2) > 1$. If, on the contrary, $\alpha_1 < \alpha_2$, then

$$\begin{aligned}(1 - 2\alpha_1)\|t_1 - m_{23}\| &= \big\|(1 + \alpha_2 - \alpha_1)(t_1 - x) + (\alpha_2 - \alpha_1)(x - t_3)\big\|\\ &\geqslant (1 + \alpha_2 - \alpha_1)\lambda - (\alpha_2 - \alpha_1)\lambda = \lambda,\end{aligned}$$

and we also obtain $\|t_1 - m_{23}\| > 1$. Therefore, $T(t_1, t_2, t_3)$ is norm-acute at t_1. In the same manner we can see that $T(t_1, t_2, t_3)$ is norm-acute at t_2 and at t_3. That $\mathcal{C}(x, \lambda)$ is the only circumcircle of $T(t_1, t_2, t_3)$ follows from [1, Theorem 4.2(b)]. $\square$

The next lemma has a very cumbersome proof, but it is the key for the proofs of Theorems 6.4 and 6.5.

Lemma 6.1. *Let $\mathcal{C}$ be the unit circle of $(\mathbb{R}^2, \|\cdot\|)$, and let $u, v \in \mathcal{C}$, $u \neq \pm v$. Let $w = \alpha(u + v) + \beta(u - v)$, $\alpha, \beta \in \mathbb{R}$.*

(i) *If $\|w - u\| = \|w - v\| = 1$, then (α, β) is equal to $(\alpha, 0)$, $(0, \beta)$ or $(1, \beta)$.*
(ii) *If $\|w - u\| < 1$ and $\|w - v\| < 1$, then $\alpha \geqslant 0$.*
(iii) *If $\|w - u\| = 1$ and $\|w - v\| < 1$, then $0 < \alpha < 1$ and $\beta < 0$.*

Proof. Consider the following identities:

$$w - u = (\alpha + \beta - 1)u + (\alpha - \beta)v, \tag{6}$$
$$w - v = (\alpha + \beta)u + (\alpha - \beta - 1)v, \tag{7}$$
$$(1 + \beta - \alpha)(w - u) = (2\alpha - 1)u + (\beta - \alpha)(w - v), \tag{8}$$
$$(\alpha + \beta)(w - u) = (2\alpha - 1)v + (\beta + \alpha - 1)(w - v). \tag{9}$$

(i) From (6) it follows that $1 = \|w - u\| \geqslant |\alpha + \beta - 1| - |\alpha - \beta| \geqslant 1 - \alpha - \beta - |\alpha - \beta|$, which gives $|\alpha - \beta| \geqslant -\alpha - \beta$, and then $\alpha \geqslant 0$ or $\beta \geqslant 0$. Similarly, from (7) it follows that $1 \geqslant |\alpha - \beta - 1| - |\alpha + \beta| \geqslant 1 - \alpha + \beta - |\alpha + \beta|$, which gives $|\alpha + \beta| \geqslant \beta - \alpha$, and then $\alpha \geqslant 0$ or $\beta \leqslant 0$. Therefore, if $\alpha < 0$, then $\beta = 0$.

Assume that $\alpha \geqslant 0$. Then

$$(6) \ \Rightarrow\ 1 \geqslant |\alpha - \beta| - |\alpha + \beta - 1| \geqslant \alpha - \beta - |\alpha + \beta - 1| \ \Rightarrow\ 1 - \alpha + \beta \geqslant -|\alpha + \beta - 1|,$$

from which it follows that either $\beta \geqslant 0$ or $\alpha \leqslant 1$. Moreover,

$$(7) \ \Rightarrow\ 1 \geqslant |\alpha + \beta| - |\alpha - \beta - 1| \geqslant \alpha + \beta - |\alpha - \beta - 1| \ \Rightarrow\ 1 - \alpha - \beta \geqslant -|\alpha - \beta - 1|,$$

from which it follows that either $\beta \leqslant 0$ or $\alpha \leqslant 1$. Therefore, if $\alpha > 1$, then $\beta = 0$.

Assume that $0 < \alpha \leqslant \frac{1}{2}$. If $\beta \leqslant -\alpha$, then

$$(9) \ \Rightarrow\ -\alpha - \beta = |\alpha + \beta| \geqslant |\beta + \alpha - 1| - |2\alpha - 1| = -\beta - \alpha + 1 - (1 - 2\alpha) \ \Rightarrow\ \alpha \leqslant 0,$$

against the hypothesis. If $\alpha \leqslant \beta$, then

$$(8) \ \Rightarrow\ 1 + \beta - \alpha = |1 + \beta - \alpha| \leqslant |2\alpha - 1| + |\beta - \alpha| = 1 - 2\alpha + \beta - \alpha \ \Rightarrow\ \alpha \leqslant 0,$$

against the hypothesis. If $-\alpha \leqslant \beta \leqslant \alpha$, then $\beta \leqslant \frac{1}{2}$ and

$$(6) \ \Rightarrow\ 1 \leqslant |\alpha + \beta - 1| + |\alpha - \beta| = 1 - \alpha - \beta + \alpha - \beta \ \Rightarrow\ \beta \leqslant 0,$$

and

$$(7) \Rightarrow 1 \leqslant |\alpha+\beta| + |\alpha-\beta-1| = \alpha+\beta-\alpha+\beta+1 \Rightarrow \beta \geqslant 0.$$

Therefore, $\beta = 0$. Similarly, if $\frac{1}{2} \leqslant \alpha < 1$, we obtain also that $\beta = 0$.

(ii) If $\alpha < 0$ and $\beta \geqslant 0$, then

$$(7) \Rightarrow 1 > \|w-v\| \geqslant |\alpha-\beta-1| - |\alpha+\beta| = -\alpha+\beta+1-|\alpha+\beta| \Rightarrow \beta-\alpha < |\alpha+\beta|,$$

which implies that $\alpha > 0$ or $\beta < 0$, against the assumption. If $\alpha < 0$ and $\beta \leqslant 0$, then

$$(6) \Rightarrow 1 > \|w-u\| \geqslant |\alpha+\beta-1| - |\alpha-\beta| = -\alpha-\beta+1-|\alpha-\beta| \Rightarrow |\alpha-\beta| > -\alpha-\beta,$$

which implies that $\alpha > 0$ or $\beta > 0$, against the assumption. Therefore, $\alpha \geqslant 0$.

(iii) We will see first that $\alpha > 0$. Assume, on the contrary, that $\alpha \leqslant 0$. If $\beta \geqslant 0$, then

$$(7) \Rightarrow 1 > \|w-v\| \geqslant |\alpha-\beta-1| - |\alpha+\beta| \geqslant 1-\alpha+\beta-|\alpha+\beta| \Rightarrow |\alpha+\beta| > \beta-\alpha,$$

which implies $\alpha > 0$ or $\beta < 0$, against the assumption. Therefore $\beta < 0$. If $\alpha \leqslant \beta$, then

$$(6) \Rightarrow 1 \geqslant |\alpha+\beta-1| - |\alpha-\beta| = 1-\alpha-\beta-(\beta-\alpha) = 1-2\beta \Rightarrow \beta \geqslant 0,$$

and we get a contradiction. If $\alpha - 1 \leqslant \beta < \alpha$, then

$$\begin{aligned}(8) \Rightarrow 1+\beta-\alpha = |1+\beta-\alpha| &\geqslant |2\alpha-1| - |\beta-\alpha|\|w-v\| \\ &\geqslant 1-2\alpha-(\alpha-\beta)\|w-v\| > 1-2\alpha-(\alpha-\beta) \Rightarrow \alpha > 0,\end{aligned}$$

against the assumption. If $\beta < \alpha - 1$, then

$$(7) \Rightarrow 1 > \|w-v\| \geqslant |\alpha+\beta| - |\alpha-\beta-1| = -\alpha-\beta-(\alpha-\beta-1) = 1-2\alpha \Rightarrow \alpha > 0,$$

against the assumption. Therefore $\alpha > 0$. Now we consider several cases:

Case 1. Assume that $\alpha+\beta \geqslant 0$ and $\alpha-\beta \geqslant 0$.

1.1: Assume that $\alpha+\beta < 1$. Then

$$(6) \Rightarrow 1 \leqslant |\alpha+\beta-1| + |\alpha-\beta| = 1-\alpha-\beta+\alpha-\beta = 1-2\beta \Rightarrow \beta \leqslant 0,$$

and

$$\begin{aligned}(9) \Rightarrow \alpha+\beta = |\alpha+\beta| &\geqslant |2\alpha-1| - |\beta+\alpha-1|\|w-v\| = |2\alpha-1| - (1-\beta-\alpha)\|w-v\| \\ &> |2\alpha-1| - (1-\beta-\alpha) \geqslant 2\alpha-1-(1-\beta-\alpha) \Rightarrow \alpha < 1.\end{aligned}$$

Moreover, assume that $\beta = 0$. If $0 < \alpha \leqslant \frac{1}{2}$, then

$$(8) \Rightarrow 1-\alpha \leqslant |2\alpha-1| + \alpha\|w-v\| = 1-2\alpha+\alpha\|w-v\| < 1-2\alpha+\alpha = 1-\alpha,$$

which is absurd. If $\frac{1}{2} < \alpha < 1$, then

$$(9) \Rightarrow \alpha \leqslant |2\alpha-1| + (1-\alpha)\|w-v\| = 2\alpha-1+(1-\alpha)\|w-v\| < \alpha,$$

which is also absurd. Therefore, $\beta < 0$.

1.2: Assume that $\alpha+\beta \geqslant 1$. Then

$$(6) \Rightarrow 1 \leqslant |\alpha+\beta-1| + |\alpha-\beta| = \alpha+\beta-1+\alpha-\beta = 2\alpha-1 \Rightarrow \alpha \geqslant 1.$$

If $\alpha - \beta \leqslant 1$, then

$$(7) \Rightarrow 1 > \|w-v\| \geqslant |\alpha+\beta| - |\alpha-\beta-1| = \alpha+\beta-(1+\beta-\alpha) = 2\alpha-1 \Rightarrow \alpha < 1,$$

and we get a contradiction. If $\alpha - \beta > 1$, then

$$(7) \Rightarrow 1 > \|w-v\| \geqslant |\alpha+\beta| - |\alpha-\beta-1| = \alpha+\beta-(\alpha-\beta-1) = 1+2\beta \Rightarrow \beta < 0,$$

and

$$(6) \Rightarrow 1 \geqslant |\alpha-\beta| - |\alpha+\beta-1| = \alpha-\beta-(\alpha+\beta-1) = 1-2\beta \Rightarrow \beta \geqslant 0,$$

and we get a contradiction. Therefore, Case 1.2 is not possible.

Case 2. Assume that $\alpha+\beta \geqslant 0$ and $\alpha-\beta<0$. Then $\beta>\alpha>0$. If $0<\alpha \leqslant \frac{1}{2}$, then

$$\begin{aligned}(8) \Rightarrow\ 1+\beta-\alpha &= |1+\beta-\alpha| \leqslant |2\alpha-1|+|\beta-\alpha|\|w-v\| \\ &= 1-2\alpha+(\beta-\alpha)\|w-v\| < 1-2\alpha+\beta-\alpha \ \Rightarrow\ \alpha<0,\end{aligned}$$

and we get a contradiction. If $\frac{1}{2}<\alpha \leqslant 1$, then

$$\begin{aligned}(8) \Rightarrow\ 1+\beta-\alpha &\leqslant |2\alpha-1|+|\beta-\alpha|\|w-v\| = 2\alpha-1+(\beta-\alpha)\|w-v\| \\ &< 2\alpha-1+\beta-\alpha \ \Rightarrow\ \alpha>1,\end{aligned}$$

against the assumption. If $\alpha>1$, then

$$(6) \Rightarrow\ 1 \geqslant |\alpha+\beta-1|-|\alpha-\beta| = \alpha+\beta-1-(\beta-\alpha) = 2\alpha-1 \ \Rightarrow\ \alpha \leqslant 1,$$

against the assumption. Therefore Case 2 is not possible.

Case 3. Assume that $\alpha+\beta<0$. Then

$$\begin{aligned}(7) \Rightarrow\ 1 > \|w-v\| &\geqslant |\alpha-\beta-1|-|\alpha+\beta| = |\alpha-\beta-1|-(-\alpha-\beta) \\ &\geqslant \alpha-\beta-1-(-\alpha-\beta) = 2\alpha-1,\end{aligned}$$

which implies that $\alpha<1$. $\square$

Theorem 6.4. *Let $T(t_1,t_2,t_3)$ be a norm-acute triangle in $(\mathbb{R}^2,\|\cdot\|)$, and let $\mathcal{C}(x,\lambda)$ be a circle through t_1,t_2,t_3. Then $\mathcal{D}(x,\lambda)$ is a minimal enclosing disc of t_1,t_2,t_3.*

Proof. Assume that $t_i \in \mathcal{C}(x,\lambda)$, $i=1,2,3$. We can assume, without loss of generality, that $\lambda=1$. Assume that there exists $\mathcal{D}(y,\mu)$, with $0<\mu<1$, such that $t_i \in \mathcal{D}(y,\mu)$, $i=1,2,3$. We will get an absurdity. From Theorem 6.3 we know that $x \in K_0(t_1,t_2,t_3)\setminus\{m_{12},m_{13},m_{23}\}$, i.e., $x=m_{13}+\alpha_x(m_{23}-m_{13})+\beta_x(m_{12}-m_{13})$, with $0 \leqslant \alpha_x<1$, $0 \leqslant \beta_x<1$, $0<\alpha_x+\beta_x \leqslant 1$. Let $\alpha_y,\beta_y \in \mathbb{R}$ be such that $y=m_{13}+\alpha_y(m_{23}-m_{13})+\beta_y(m_{12}-m_{13})$. Let

$$\begin{aligned}&u_1=t_2-x, \qquad u_2=t_2-x, \qquad u_3=t_3-x, \\ &v_1=t_3-x, \qquad v_2=t_1-x, \qquad v_3=t_1-x.\end{aligned}$$

Then, $\|u_i\|=\|v_i\|=1$, $u_i \neq \pm v_i$, $i=1,2,3$. Moreover, $w:=y-x=\alpha_i(u_i+v_i)+\beta_i(u_i-v_i)$, $i=1,2,3$, where

$$\begin{aligned}&\alpha_1=\frac{\alpha_y-\alpha_x}{2(1-\alpha_x)}, \qquad \alpha_2=\frac{\beta_y-\beta_x}{2(1-\beta_x)}, \qquad \alpha_3=\frac{\alpha_x-\alpha_y+\beta_x-\beta_y}{2(\alpha_x+\beta_x)}, \\ &\beta_1=\frac{(\alpha_y-1)\beta_x+(1-\alpha_x)\beta_y}{2(1-\alpha_x)}, \qquad \beta_2=\frac{(1-\beta_x)\alpha_y+(\beta_y-1)\alpha_x}{2(1-\beta_x)}, \qquad \beta_3=\frac{\alpha_y\beta_x-\alpha_x\beta_y}{2(\alpha_x+\beta_x)}.\end{aligned}$$

Since $\|w-u_i\| \leqslant \mu<1$, $\|w-v_i\| \leqslant \mu<1$, $i=1,2,3$, it follows from Lemma 6.1(ii) that $\alpha_i \geqslant 0$, $i=1,2,3$. But this implies that $\alpha_x=\alpha_y$ and $\beta_x=\beta_y$, and then $x=y$, which is absurd. $\square$

Our next theorem shows that there exist triangles in normed planes which have, surprisingly, a unique circumcircle, but infinitely many minimal enclosing discs (see Fig. 9).

Theorem 6.5. *Let $T(t_1,t_2,t_3)$ be a norm-acute triangle in $(\mathbb{R}^2,\|\cdot\|)$ such that there exists exactly one circle through t_1, t_2, t_3. To simplify, we assume that this circle is $\mathcal{C}(o,1)$.*

(i) *If there are no indices $i,j \in \{1,2,3\}$ such that $[t_i,-t_j]$ is in the relative interior of a segment from $\mathcal{C}(o,1)$, then $\mathcal{D}(o,1)$ is the unique minimal enclosing disc of t_1,t_2,t_3, and then*

$$MEDC(t_1,t_2,t_3)=\{o\}=\mathcal{G}(t_1,1)\cap\mathcal{G}(t_2,1)\cap\mathcal{G}(t_3,1),$$

where "$\mathcal{G}$" means, indistinctly, "$\mathcal{C}$" or "$\mathcal{D}$".

(ii) *Assume that $[t_1,-t_2]$ is in the relative interior of a segment $[t_1',-t_2']$ from $\mathcal{C}(o,1)$, that we assume to be maximal, with $t_1',t_1,-t_2,-t_2'$ aligned in that order. Then*

$$MEDC(t_1,t_2,t_3)=\mathcal{C}(t_1,1)\cap\mathcal{C}(t_2,1)\cap\mathcal{D}(t_3,1)=\left[o,t_1-t_1'\right]\cap\left[o,t_2-t_2'\right].$$

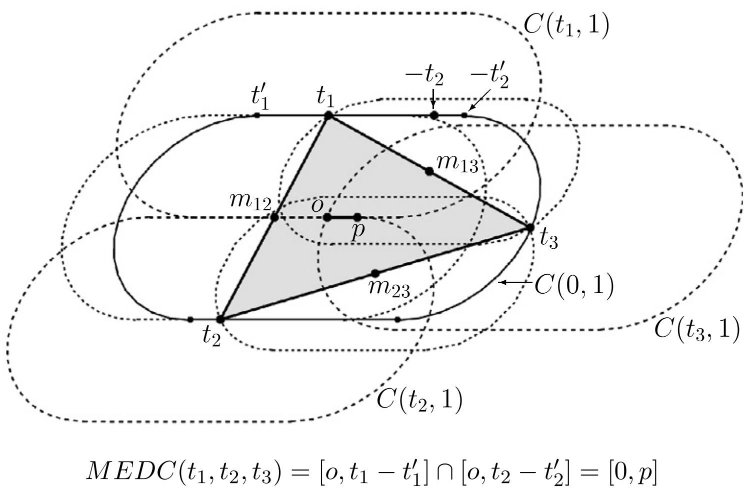


Fig. 9. Locus of the centers of the minimal enclosing discs of a norm-acute triangle.

Proof. By Theorems 6.1 and 6.4 we know that there exists a circle $\mathcal{C}(x,\lambda)$ through t_1,t_2,t_3, and that $\mathcal{D}(x,\lambda)$ is a minimal enclosing disc. By Theorem 6.3 we also know that $x\in K_0(t_1,t_2,t_3)\setminus\{m_{12},m_{13},m_{23}\}$. We will assume, without loss of generality, that $x=o$ and $\lambda=1$. Then

$$o=\alpha m_{12}+\beta m_{13}+(1-\alpha-\beta)m_{23}, \tag{10}$$

with $0\leqslant\alpha<1$, $0\leqslant\beta<1$ and $0<\alpha+\beta\leqslant 1$.

Also observe that if $\mathcal{D}(y,1)$ is a minimal enclosing disc of the points t_1,t_2,t_3, then at least two of the points belong to $\mathcal{C}(y,1)$. Assume, on the contrary, that t_1 and t_2 are interior points of $\mathcal{D}(y,1)$. Then there exists $\varepsilon>0$ such that $\mathcal{D}(t_1,\varepsilon)\cup\mathcal{D}(t_2,\varepsilon)\subset\mathcal{D}(y,1)$. Thus, we can move t_1 and t_2 in any direction inside of $\mathcal{D}(t_1,\varepsilon)$ and $\mathcal{D}(t_2,\varepsilon)$, respectively, without going out of $\mathcal{D}(y,1)$. This allows to obtain $y'\in(y,t_3]$ such that t_1,t_2,t_3 are interior points of $\mathcal{D}(y',1)$, and then 1 is not the radius of the minimal enclosing disc.

(i) Assume that there exists a minimal enclosing disc of t_1,t_2,t_3 different from $\mathcal{D}(o,1)$, say $\mathcal{D}(y,1)$, $y\neq o$. Assume that $t_1,t_2\in\mathcal{C}(y,1)$, i.e., $\|t_1-y\|=\|t_2-y\|=1$. Since $\mathcal{C}(o,1)$ is the unique circle through t_1,t_2,t_3, we have that $\|t_3-y\|<1$. Since t_1+t_2 and t_1-t_2 are linearly independent, it follows from Lemma 6.1(i) that

$$y=\alpha_3(t_1+t_2)+\beta_3(t_1-t_2), \tag{11}$$

with (α_3,β_3) equal to $(\alpha_3,0)$, $(0,\beta_3)$ or $(1,\beta_3)$. We will see that only the first case is possible, i.e., $\beta_3=0$, and also that $\alpha_3<0$.

From (10) it follows that

$$t_2=\left(\frac{\alpha+\beta}{\beta-1}\right)t_1+\left(\frac{1-\alpha}{\beta-1}\right)t_3.$$

Substituting t_2 in (11), we get that

$$y=\alpha_1(t_2+t_3)+\beta_1(t_2-t_3)=\alpha_2(t_1+t_3)+\beta_2(t_1-t_3),$$

where

$$\alpha_1=\frac{\alpha_3(\alpha+\beta-1)-\beta_3}{\alpha+\beta},\qquad \beta_1=\frac{\beta\alpha_3-\alpha\beta_3}{\alpha+\beta},\qquad \alpha_2=\frac{\beta_3-\beta\alpha_3}{1-\beta},\qquad \beta_2=\frac{\alpha_3(1-\alpha-\beta)+\alpha\beta_3}{1-\beta}.$$

From Lemma 6.1(iii) it follows that

$$0<\alpha_1<1,\qquad \beta_1<0,\qquad 0<\alpha_2<1,\qquad \beta_2<0. \tag{12}$$

Assume that $\alpha_3=0$. Then $\alpha_1=\frac{-\beta_3}{\alpha+\beta}$, and from (12) it follows that $\beta_3<0$. Moreover, $\alpha_2=\frac{\beta_3}{1-\beta}$, and again from (12) we get that $\beta_3>0$, which is absurd. Assume that $\alpha_3=1$. Then $\alpha_1=\frac{\alpha+\beta-1-\beta_3}{\alpha+\beta}>0$, from which it follows that $\beta_3<\alpha+\beta-1\leqslant 0$. Moreover, since $\beta_1=\frac{\beta-\alpha\beta_3}{\alpha+\beta}<0$, we have that $\beta<\alpha\beta_3\leqslant 0$, which is absurd. Therefore, the only possibility is that $(\alpha_3,\beta_3)=(\alpha_3,0)$, i.e., $\beta_3=0$. Then, from (12) we have that $0<\alpha_1=\frac{\alpha_3(\alpha+\beta-1)}{\alpha+\beta}$, which implies that $\alpha_3<0$. Thus $y=\alpha_3(t_1+t_2)$, and taking $\mu=\frac{1-\alpha_3}{1-2\alpha_3}$ we have that $0<\mu<1$, and

$$t_1 = \mu(t_1 - y) + (1-\mu)(y - t_2),$$
$$-t_2 = \mu(y - t_2) + (1-\mu)(t_1 - y),$$

which implies that $[t_1, -t_2]$ is in the interior of the segment $[t_1 - y, y - t_2]$ that is contained in $\mathcal{C}(o,1)$. The second identity in (i) follows trivially.

(ii) Assume that $[t_1, -t_2]$ is in the relative interior of the segment $[t_1', -t_2']$ from $\mathcal{C}(o,1)$. Let $y \in MEDC(t_1, t_2, t_3)$. If $y = o$, then it is clear that $y \in \mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1) \cap \mathcal{D}(t_3,1)$. Thus, assume that $y \neq o$. Then $\|t_i - y\| = \|t_j - y\| = 1 > \|t_k - y\|$, with $\{i,j,k\} = \{1,2,3\}$. As in the proof of (i), it follows that $[t_i, -t_j]$ is in the relative interior of a segment of $\mathcal{C}(o,1)$. Since, by hypothesis, the same occurs with $[t_1, -t_2]$, necessarily, $\{i,j\} = \{1,2\}$ and $k = 3$. Therefore, $\|t_1 - y\| = \|t_2 - y\| = 1$ and $\|t_3 - y\| < 1$, i.e., $y \in \mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1) \cap \mathcal{D}(t_3,1)$. Conversely, if $y \in \mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1) \cap \mathcal{D}(t_3,1)$, then it is clear that $y \in MEDC(t_1, t_2, t_3)$.

We will see now that

$$\mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1) \cap \mathcal{D}(t_3,1) = \left[o, t_1 - t_1'\right] \cap \left[o, t_2 - t_2'\right]. \tag{13}$$

By hypothesis,

$$t_1 = (1-\mu_1)t_1' + \mu_1\left(-t_2'\right), \qquad -t_2 = (1-\mu_2)t_1' + \mu_2\left(-t_2'\right), \tag{14}$$

with $0 < \mu_1 < \mu_2 < 1$. Let $z \in \mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1) \cap \mathcal{D}(t_3,1)$, and assume that $z \neq o$. Then $\|t_1 - z\| = \|t_2 - z\| = 1$ and $\|t_3 - z\| < 1$, because we are assuming that the circle through t_1, t_2, t_3 is unique. As in the proof of (i), we get that $z = \bar{\alpha}_3(t_1 + t_2)$, with $\bar{\alpha}_3 < 0$. From (14) it follows that

$$t_1 - z = (1-\bar{\mu}_1)t_1' + \bar{\mu}_1\left(-t_2'\right), \qquad z - t_2 = (1-\bar{\mu}_2)t_1' + \bar{\mu}_2\left(-t_2'\right),$$

with

$$\bar{\mu}_1 = \mu_1 + \bar{\alpha}_3(\mu_2 - \mu_1), \qquad \bar{\mu}_2 = \mu_2 + \bar{\alpha}_3(\mu_1 - \mu_2).$$

Since $[t_1', -t_2']$ is maximal, we have that $0 \leqslant \bar{\mu}_1 \leqslant 1$ and $0 \leqslant \bar{\mu}_2 \leqslant 1$, from which follows that

$$0 < \frac{-\bar{\alpha}_3(\mu_2 - \mu_1)}{\mu_1} \leqslant 1, \qquad 0 < \frac{-\bar{\alpha}_3(\mu_2 - \mu_1)}{1 - \mu_2} \leqslant 1.$$

Finally, since

$$z = \left(\frac{-\bar{\alpha}_3(\mu_2 - \mu_1)}{\mu_1}\right)\left(t_1 - t_1'\right) = \left(\frac{-\bar{\alpha}_3(\mu_2 - \mu_1)}{1 - \mu_2}\right)\left(t_2 - t_2'\right),$$

we obtain that $z \in [o, t_1 - t_1'] \cap [o, t_2 - t_2']$. Reciprocally, let $z \in [o, t_1 - t_1'] \cap [o, t_2 - t_2']$. Since $t_1 - t_1' = \mu_1(-t_1' - t_2')$ and $t_2 - t_2' = (1-\mu_2)(-t_1' - t_2')$, we have that $[o, t_1 - t_1'] \cap [o, t_2 - t_2'] = [o, \bar{\mu}(-t_1' - t_2')]$, where $\bar{\mu} = \min\{\mu_1, 1 - \mu_2\}$. Therefore, $z = \gamma\bar{\mu}(-t_1' - t_2')$, with $0 \leqslant \gamma \leqslant 1$. Moreover,

$$t_1 - z = \left(1 - \frac{\gamma\bar{\mu}}{\mu_1}\right)t_1 + \frac{\gamma\bar{\mu}}{\mu_1}t_1', \qquad z - t_2 = \left(1 - \frac{\gamma\bar{\mu}}{1-\mu_2}\right)(-t_2) + \frac{\gamma\bar{\mu}}{1-\mu_2}\left(-t_2'\right).$$

Since $0 \leqslant \frac{\gamma\bar{\mu}}{\mu_1} \leqslant 1$ and $0 \leqslant \frac{\gamma\bar{\mu}}{1-\mu_2} \leqslant 1$, we get that $z \in \mathcal{C}(t_1,1) \cap \mathcal{C}(t_2,1)$. To prove that $z \in \mathcal{D}(t_3,1)$, observe that from (10) it follows that

$$t_3 = \left(\frac{\alpha + \beta}{1 - \alpha}\right)(-t_1) + \left(\frac{1 - \beta}{1 - \alpha}\right)(-t_2),$$

and having (14) in mind, we obtain that

$$t_3 - z = \gamma_1 t_1' + \gamma_2 t_2' + (1 - \gamma_1 - \gamma_2)t_3, \tag{15}$$

where

$$\gamma_1 = \frac{\gamma\bar{\mu}(1 - \alpha - \beta)}{\beta(1-\mu_1) + \alpha(\mu_2 - \mu_1) + \mu_2(1 - \alpha - \beta)},$$
$$\gamma_2 = \frac{\gamma\bar{\mu}\beta}{\beta(1-\mu_1) + \alpha(\mu_2 - \mu_1) + \mu_2(1 - \alpha - \beta)}.$$

Since $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$, and

$$\begin{aligned}\gamma_1+\gamma_2 &= \frac{\gamma\bar{\mu}(1-\alpha)}{\beta(1-\mu_1)+\alpha(\mu_2-\mu_1)+\mu_2(1-\alpha-\beta)}\\ &= \frac{\gamma\bar{\mu}(1-\alpha)}{\beta(1-\mu_2)-(\alpha+\beta)\mu_1+\mu_2} \leqslant \frac{\gamma\bar{\mu}(1-\alpha)}{\beta(1-\mu_2)-(\alpha+\beta)\mu_1+\mu_1}\\ &= \frac{\gamma\bar{\mu}(1-\alpha)}{\beta(1-\mu_2)+(1-\alpha-\beta)\mu_1} \leqslant \frac{\gamma\bar{\mu}(1-\alpha)}{\beta\bar{\mu}+(1-\alpha-\beta)\bar{\mu}} = \gamma \leqslant 1,\end{aligned}$$

it follows from (15) that $t_3 - z \in \operatorname{conv}\{t_1', t_2', t_3\} \subset \mathcal{D}(0,1)$, i.e., $z \in \mathcal{D}(t_3, 1)$. □

Now we consider the case of norm-acute triangles with more than one circumcircle.

Lemma 6.2. *Let $T(t_1,t_2,t_3)$ be a norm-acute triangle in $(\mathbb{R}^2, \|\cdot\|)$ such that more than one circle passes through t_1, t_2, t_3. Then the following properties hold*:

(i) *All circumcircles of $T(t_1,t_2,t_3)$ contain at least one side of $T(t_1,t_2,t_3)$, say $[t_1,t_2]$.*
(ii) $\|t_1-t_3\| = \|t_2-t_3\| > \|t_1-t_2\|$.
(iii) *All circumcenters of $T(t_1,t_2,t_3)$ belong to $[m_{13}, m_{23}] \setminus \{m_{13}, m_{23}\}$.*
(iv) *All circumradii of $T(t_1,t_2,t_3)$ have length $\frac{1}{2}\|t_1-t_3\|$.*
(v) *For every point $x \in [m_{13}, m_{23}]$ the equality $\|x-t_3\| = \frac{1}{2}\|t_1-t_3\|$ holds.*
(vi) *If $\mathcal{D}(w,\lambda)$ is a minimal enclosing disc of t_1,t_2,t_3, then $\lambda = \frac{1}{2}\|t_1-t_3\|$ and $\mathcal{C}(w,\lambda)$ is a circumcircle of t_1,t_2,t_3.*

Proof. Statement (i) follows from [1, Corollary 3.1]. The equality in (ii) as well as statements (iii) and (iv) follow from [1, Theorem 4.2] and Theorem 6.3. Let y and z be two different circumcenters of $T(t_1,t_2,t_3)$. By (iii), $y = \beta m_{13} + (1-\beta)m_{23}$, with $0<\beta<1$, and $z = \gamma m_{13} + (1-\gamma)m_{23}$, with $0<\gamma<1$. We can assume, without loss of generality, that $\beta > \gamma$. Then $(2+\beta-\gamma)(t_1-t_2) = 2(t_1-z)+2(y-t_2)$, and from (iv) we get that $2\|t_1-t_2\| < (2+\beta-\gamma)\|t_1-t_2\| \leqslant 2\|t_1-z\|+2\|y-t_2\| = 2\|t_1-t_3\|$, yielding the inequality in (ii). Moreover, if $x \in [m_{13}, m_{23}]$, i.e., $x = \alpha m_{13} + (1-\alpha)m_{23}$ with $0 \leqslant \alpha \leqslant 1$, then $x - t_3 = \mu t_1 + (1-\mu)t_2 - y$ with $0 < \mu := \frac{\alpha+\beta}{2} < 1$. From (i) and (iv) it follows that $\|x-t_3\| = \frac{1}{2}\|t_1-t_3\|$, i.e., (v) holds. To prove (vi), assume that $\mathcal{D}(w,\lambda)$ is a minimal enclosing disc of t_1,t_2,t_3. By Theorem 6.4 and (iv) we have $\lambda = \frac{1}{2}\|t_1-t_3\|$. Let $y = \beta m_{13} + (1-\beta)m_{23}$, with $0<\beta<1$, be a circumcenter of $T(t_1,t_2,t_3)$, and let $\bar{w} = \beta t_1 + (1-\beta)t_2$. Then $\bar{w} - t_3 = 2(y-t_3)$, and from (v) it follows that $\|\bar{w}-t_3\| = 2\lambda$. Since $\bar{w}, t_3 \in \mathcal{D}(w,\lambda)$, we have that, necessarily, $\bar{w}, t_3 \in \mathcal{C}(w,\lambda)$. But $\bar{w}$ is in the relative interior of the segment $[t_1,t_2] \subset \mathcal{D}(w,\lambda)$, which implies that $t_1, t_2 \in \mathcal{C}(w,\lambda)$. □

Lemma 6.2 says that any norm-acute triangle with more than one circumcircle is isosceles. The next theorem describes the locus of the centers of the minimal enclosing discs of such triangles.

Theorem 6.6. *Let $T(t_1,t_2,t_3)$ be a norm-acute triangle in $(\mathbb{R}^2, \|\cdot\|)$ such that more than one circle passes through t_1,t_2,t_3 and $\|t_1-t_3\| = \|t_2-t_3\|$ (see Lemma 6.2). Let $\lambda = \frac{1}{2}\|t_1-t_3\|$. Then*

$$\begin{aligned} MEDC(t_1,t_2,t_3) &= \mathcal{C}(t_1,\lambda) \cap \mathcal{C}(t_2,\lambda) \cap \langle m_{13}, m_{23}\rangle\\ &= \mathcal{G}(t_1,\lambda) \cap \mathcal{G}(t_2,\lambda) \cap \mathcal{G}(t_3,\lambda),\end{aligned}$$

where "$\mathcal{G}$" means, indistinctly, "$\mathcal{C}$" or "$\mathcal{D}$".

Proof. Let $\mathcal{D}(x,\lambda)$ be a minimal enclosing disc of $T(t_1,t_2,t_3)$. By Lemma 6.2, $\lambda = \frac{1}{2}\|t_1-t_3\|$, $\mathcal{C}(x,\lambda)$ is a circumcircle of t_1,t_2,t_3, and $x \in [m_{13}, m_{23}] \setminus \{m_{13}, m_{23}\}$. Thus we get that $MEDC(t_1,t_2,t_3) \subset \mathcal{C}(t_1,\lambda) \cap \mathcal{C}(t_2,\lambda) \cap \langle m_{13}, m_{23}\rangle$. Conversely, let $x \in \mathcal{C}(t_1,\lambda) \cap \mathcal{C}(t_2,\lambda) \cap \langle m_{13}, m_{23}\rangle$. Then $\|x-t_1\| = \|x-t_2\| = \lambda$ and $x = \alpha m_{13} + (1-\alpha)m_{23}$ with $\alpha \in \mathbb{R}$. By Theorem 6.4 and Lemma 6.2(v), to prove that $x \in MEDC(t_1,t_2,t_3)$ we only need to see that $0<\alpha<1$. Let us consider the convex function $f(\mu) = \|\mu m_{13} + (1-\mu)m_{23} - t_2\|$, $\mu \in \mathbb{R}$. Then $f(0) = \|m_{23}-t_2\| = \lambda = f(-1)$. Moreover, since $T(t_1,t_2,t_3)$ is norm-acute, we have $f(1) = \|m_{13}-t_2\| > \lambda$ and $f(-2) = \|m_{23}-t_1\| > \lambda$. Therefore, $f(\mu) > \lambda$ if $\mu \geqslant 1$ or $\mu \leqslant -2$. Since $f(\alpha) = \|x-t_2\| = \lambda = \|x-t_1\| = f(\alpha-2)$, necessarily $0<\alpha<1$. The second identity also follows from Lemma 6.2. □

The next corollary follows directly from Theorem 6.1 and Lemma 6.2.

Corollary 6.1. *Let $T(t_1,t_2,t_3)$ be a norm-acute triangle in $(\mathbb{R}^2, \|\cdot\|)$ and assume that (at least) one of the following properties hold*:

(i) *$T(t_1,t_2,t_3)$ is scalene;*
(ii) *$T(t_1,t_2,t_3)$ is equilateral;*
(iii) *$T(t_1,t_2,t_3)$ is isosceles with one side larger than the other two.*

Then $T(t_1,t_2,t_3)$ has a unique circumcircle.

7. Summary

From the above results it follows that for any triangle $T(t_1, t_2, t_3)$ the set $MEDC(t_1, t_2, t_3)$ is a segment that can degenerate to a point. The following table summarizes the different results according to the two possibilities.

$MEDC(t_1, t_2, t_3)$	Triangle type		
	Norm-obtuse or norm-right	Double norm-right	Norm-acute
A point	Corollary 4.3	Impossible by Lemma 5.1	Theorem 6.5(i)
A segment	Theorem 4.1, Theorem 4.2	Theorem 5.1	Theorem 6.5(ii), Theorem 6.6

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